Critical Reasoning 28 – Introduction to Many-Valued Logics II

After Fronhöfer (2011)

This study unit follows on from Critical Reasoning 25. Unless otherwise indicated, content is based almost exclusively on the presentation of Fronhöfer (2011) with only minor changes.

A Derivation System for Classical Propositional Logic

First Approach

We require axioms and rules

- Axioms: a certain set of good formulae which shall be true
- Rules of Inference and/or Rules of Proof: **A**, … , **B** /∴ **C** where **A**, … , **B** and **C** are formulae
- If our axioms are formulae we require one or more rules of substitution.

Disadvantages:

- Axioms and explicit substitution rules of inference make derivations longer.
- Correct rules of substitution and their correct application may be tricky. (p. 214)

Second Approach[1]

Defn. 4.1 An **axiom schema** stands for infinitely many formulae that have the overall form exemplified by the schema.

An **instance of an axiom schema** is defined as any formula that results from the uniform substitution of formulae of the language for each of the letters occurring in the axiom schema.

By **uniform substitution** we mean that in any given instance, the same formula must be substituted for every occurrence of the same letter.

Fronhöfer writes: **F**/**P** (read: **F** instead of **P**) for the uniform substitution of **P** by **F**.

(p. 214)

Axiomatic System for Classical Propositional Logic

Defn. 4.2 **CLA** (Classical propositional Logic Axiomatic system)

- Axiom Schemata

CL1 $P \supset (Q \supset P)$

[1] Compare Fronhöfer's development of a Classical Propositional Logic with that of Copi's R.S. in Critical Reasoning 21.

CL2 $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$

CL3 $(\sim P \supset \sim Q) \supset (Q \supset P)$

- Rule of Inference

Modus Ponens (M.P.)

P and $P \supset Q$ /∴ Q (p. 216)

Rem. 4.3 M.P. is also known as **separation**, **implication elimination** or the **rule of detachment** because it allows the conclusion to be "detached" from the premises.

Rem. 4.4 Each of the axiom schemata is in the form of a tautology, with P, Q and R serving as propositional variables. See the truth-table for **CL2** below.

(P	⊃	(Q	⊃	R))	⊃	((P	⊃	Q)	⊃	(P	⊃	R))
T	T	T	T	T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	T	T	F	T	F	F
T	T	F	T	T	T	T	F	F	T	T	T	T
T	T	F	T	F	T	T	F	F	T	T	F	F
F	T	T	T	T	T	F	T	T	T	F	T	T
F	T	T	F	F	T	F	T	T	T	F	T	F
F	T	F	T	T	T	F	T	F	T	F	T	T
F	T	F	T	F	T	F	T	F	T	F	T	F

CL1 and **CL2** can be verified as tautologous in the same way.

- Any instance of **CL1** to **CL3** will also be tautologous (proof by induction).
- *Modus Ponens* is truth-preserving *i.e.* if P and $P \supset Q$ are both true then so is Q.

(p. 216)

Derivations

Defn. 4.5 A **derivation** is a finite sequence of formulae, each of which is

- designated as an assumption, or
- is an instance of an axiom schema, or
- can be derived from earlier formula using *Modus Ponens*.

Convention: The formulae designated as assumptions must begin the chain of derivations. If **F** is the last formula in a chain of derivation then we speak of a derivation of **F**.

Defn. 4.6 A formula **F** of Ł$_3$ is **derivable** from a set of formulae $\boldsymbol{S}$ (symbolised $\boldsymbol{S} \vdash$ **F**) iff there is a derivation of **F** such that all assumptions in the derivation are elements of $\boldsymbol{S}$.

A formula **F** is **provable** (symbolised ⊢ **F**) iff **F** is derivable from the empty set (∅).

***E.g.* 4.7** In the following example of a derivation, Fronhöfer uses the method of Conditional Proof but does not explicitly discharge each assumption. Notwithstanding, the same rules for Conditional Poof as in Critical Reasoning 09 apply.

1.	F	Conditional Proof Assumption (CPA)
2.	(G ⊃ F) ⊃ (F ⊃ H)	CPA
3.	F ⊃ (G ⊃ F)	CL1, F/P, G/Q
4.	G ⊃ F	1,3 M.P.
5.	F ⊃ H	2,4 M.P.
6.	H	1,5 M.P. (p. 218)

***E.g.* 4.8** The following example shows that $A \supset B$ is derivable from $\sim A$.

1.	∼A	CPA
2.	∼A ⊃ (∼B ⊃ ∼A)	CL1, ∼A/P, ∼B/Q
3.	∼B ⊃ ∼A	1,2 M.P.
4.	(∼B ⊃ ∼A) ⊃ (A ⊃ B)	CL3, B/P, A/Q
5.	A ⊃ B	3,4 M.P. (p. 218)

Lem. 4.9 For a deductive system consisting of axiom schemata and M.P., the following hold:

a) If $\boldsymbol{S} \vdash$ **A**, then $\boldsymbol{S}' \vdash$ **A** for every superset $\boldsymbol{S}'$ of $\boldsymbol{S}$.
b) Hence, in particular, if $\vdash$ **A**, then $\boldsymbol{S} \vdash$ **A** for every set of formulae $\boldsymbol{S}$.
c) Hence, in particular, if **A** is an axiom, then $\boldsymbol{S} \vdash$ **A** for every set of formulae $\boldsymbol{S}$.
d) If $\boldsymbol{S} \vdash$ **A**, then there exists a finite subset $\boldsymbol{S}'$ of $\boldsymbol{S}$ such that $\boldsymbol{S}' \vdash$ **A**.
e) If **A** is an element of $\boldsymbol{S}$, then $\boldsymbol{S} \vdash$ **A**.
f) If $\boldsymbol{S} \vdash$ **A** and $\boldsymbol{S} \vdash$ **A** ⊃ **B**, then $\boldsymbol{S} \vdash$ **B**.

Proof: The first five points can be trivially proved from defn. 4.5 (derivation) and defn. 4.6 (derivability). To prove the last point requires that we concatenate the derivations of **A** and **A** ⊃ **B**, make the assumptions at the begging and then apply M.P.

Defn. 4.10 A derivation that does not contain any assumptions is called a **proof**.

A formula **F** is called a **theorem** if there is a proof ending in **F**. We call this proof a "proof of the theorem **F**".

***E.g.* 4.11** The following proof establishes that $A \supset A$ is a theorem.

1.	A ⊃ ((A ⊃ A) ⊃ A)	CL1, A/P, A ⊃ A/Q
2.	(A ⊃ ((A ⊃ A) ⊃ A)) ⊃ ((A ⊃ (A ⊃ A)) ⊃ (A ⊃ A))	CL2, A/P, A ⊃ A/Q, A/R
3.	(A ⊃ (A ⊃ A)) ⊃ (A ⊃ A)	1,2 M.P

4. $A \supset (A \supset A)$ CL1, A/P, A/Q

5. $A \supset A$ 3,4 M.P (p. 220)

Consistency

Defn. 4.12 In Critical Reasoning 16 we introduced the Post criterion for consistency according to which: "Any system is consistent if it contains (that is, can express) a formula that is not provable as a theorem within the system." According to Fronhöfer's definitions:

- A set of formulae $\mathcal{S}$ is **syntactically consistent** iff there does *not* exist a formula **F** such that both **F** and ~**F** are provable from $\mathcal{S}$.
- A set of formulae $\mathcal{S}$ is **syntactically inconsistent** iff there *does* exist a formula **F** such that both **F** and ~**F** are provable from $\mathcal{S}$.
- A set of formulae $\mathcal{S}$ is **maximally consistent** iff $\mathcal{S}$ is consistent *and* $\mathcal{S} \vdash$ **F** for any formula **F** such that $\mathcal{S} \cup \{$**F**$\}$ is syntactically consistent.

Recall that because the presence of a contradiction (**F** • ~**F**) within a system would render every proposition a provable theorem, Post's criterion for consistency and Fronhöfer's definitions excludes such systems from the class of consistent systems.

Defn. 4.13 A set of formulae $\mathcal{S}$ is **semantically consistent** or **satisfiable** iff there exists an interpretation on which all formulae in $\mathcal{S}$ evaluate to T. (p. 222)

Soundness

Defn. 4.14 A derivation system for classical propositional logic is said to be **sound** iff

- All theorems are tautologies, and
- whenever a formula **F** is derivable from a set of formulae $\mathcal{S}$, then **F** is also entailed by $\mathcal{S}$.

Lem. 4.15 The **CLA** system is sound.

Proof: CLA is a sound derivation system because all its axioms are tautologies and its single rule of inference (M.P.) is truth-preserving. (*cf.* Rem 4.4) (p. 222)

Completeness

Defn. 4.16 Recall that we introduced the concepts of expressive and deductive completeness in Critical Reasoning 16 by way of discussion. Here Fronhöfer is more succinct:

- A derivation system for classical propositional logic is said to be **weakly complete** iff every tautology of classical logic is a theorem within the system.

- A derivation system is said to be **strongly complete** (or just **complete**) iff, in addition, whenever a set of formulae $\boldsymbol{S}$ entails a formula **F**, then **F** is also derivable from $\boldsymbol{S}$ within the system.
- A derivation system is said to be **adequate** for classical propositional logic if it is both sound and complete.

CLA is both complete and sound for classical propositional logic.

Thrm. 4.17 For a formula **A** of **CL** and a set of formulae $\boldsymbol{S}$ of **CL**, the following obtain

- if $\boldsymbol{S} \vDash$ **A**, then $\boldsymbol{S} \vdash$ **A** (**Strong Completeness Theorem**)
- if $\vDash$ **A**, then $\vdash$ **A** (**Weak Completeness Theorem**) (p. 224)

To derive a tautology like $F \text{ v } (F \supset G)$ in **CLA**, we must first rewrite any formulae containing •, v and ≡ which do not feature on the axiom scheme of **CLA**, in terms of only ∼ and ⊃ which do.

Defn. 4.18 $P \text{ v } Q := {\sim}P \supset Q$

$P \bullet Q := {\sim}(P \supset {\sim}Q)$

$P \equiv Q := (P \supset Q) \bullet (Q \supset P)$ which is equivalent to ${\sim}((P \supset Q) \supset {\sim}(Q \supset P))$ (p. 224)

***E.g.* 4.19** To show that $F \text{ v } (F \supset G)$ is a theorem of **CLA**, we first rewrite the formula as ${\sim}F \supset (F \supset G)$ using the definition for disjunction above, then we construct a derivation for the latter using only CL1 to CL3 and M.P. Thus:

1. $(({\sim}F \supset (({\sim}G \supset {\sim}F) \supset (F \supset G))) \supset$
$(({\sim}F \supset ({\sim}G \supset {\sim}F)) \supset ({\sim}F \supset (F \supset G)))) \supset$
$((({\sim}G \supset {\sim}F) \supset (F \supset G)) \supset$
$(({\sim}F \supset (({\sim}G \supset {\sim}F) \supset (F \supset G))) \supset$
$(({\sim}F \supset ({\sim}G \supset {\sim}F)) \supset$
$({\sim}F \supset (F \supset G)))))$
— CL1, $({\sim}F \supset (({\sim}G \supset {\sim}F) \supset (F \supset G))) \supset (({\sim}F \supset ({\sim}G \supset {\sim}F)) \supset ({\sim}F \supset (F \supset G))/P$
$({\sim}G \supset {\sim}F) \supset (F \supset G)/Q$

2. $({\sim}F \supset (({\sim}G \supset {\sim}F) \supset (F \supset G)) \supset$
$(({\sim}F \supset ({\sim}G \supset {\sim}F)) \supset ({\sim}F \supset (F \supset G)))$
— CL2, ${\sim}F/P$, ${\sim}G \supset {\sim}F/Q$, $F \supset G/R$

3. $(({\sim}G \supset {\sim}F) \supset (F \supset G)) \supset$
$(({\sim}F \supset (({\sim}G \supset {\sim}F) \supset (F \supset G))) \supset$
$(({\sim}F \supset ({\sim}G \supset {\sim}F)) \supset ({\sim}F \supset (F \supset G))))$
— 1,2 M.P

4. $((({\sim}G \supset {\sim}F) \supset (F \supset G)) \supset$
$(({\sim}F \supset (({\sim}G \supset {\sim}F) \supset (F \supset G))) \supset$
$(({\sim}F \supset ({\sim}G \supset {\sim}F) \supset$
$({\sim}F \supset (F \supset G))))) \supset$
$(((({\sim}G \supset {\sim}F) \supset (F \supset G)) \supset$
$({\sim}F \supset (({\sim}G \supset {\sim}F) \supset (F \supset G)))) \supset$
— CL2, $({\sim}G \supset {\sim}F) \supset (F \supset G)/P$
${\sim}F \supset (({\sim}G \supset {\sim}F) \supset (F \supset G))/Q$
$({\sim}F \supset ({\sim}G \supset {\sim}F)) \supset ({\sim}F \supset (F \supset G))/R$

$(((\sim G \supset \sim F) \supset (F \supset G)) \supset$
$((\sim F \supset (\sim G \supset \sim F)) \supset$
$(\sim F \supset (F \supset G)))))$

5. $(((\sim G \supset \sim F) \supset (F \supset G)) \supset$ 3,4 M.P.
$(\sim F \supset ((\sim G \supset \sim F) \supset (F \supset G)))) \supset$
$(((\sim G \supset \sim F) \supset (F \supset G)) \supset$
$((\sim F \supset (\sim G \supset \sim F)) \supset$
$(\sim F \supset (F \supset G))))$

6. $((\sim G \supset \sim F) \supset (F \supset G)) \supset$ CL1, $\sim F/Q$
$(\sim F \supset ((\sim G \supset \sim F) \supset (F \supset G)))$ $(\sim G \supset \sim F) \supset (F \supset G)/P$

7. $((\sim G \supset \sim F) \supset (F \supset G)) \supset$ 5,6 M.P.
$((\sim F \supset (\sim G \supset \sim F)) \supset$
$(\sim F \supset (F \supset G)))$

8. $\sim F \supset (\sim G \supset \sim F)$ CL1, $\sim F/P$, $\sim G/Q$

9. $(\sim G \supset \sim F) \supset (F \supset G)$ CL3, G/P, F/Q

10. $(\sim F \supset (\sim G \supset \sim F)) \supset (\sim F \supset (F \supset G))$ 9,7 M.P.

11. $\sim F \supset (F \supset G)$ 8,10 M.P.

We may therefore conclude that F v $(F \supset G)$ is a theorem of **CLA**. (p. 226)

Derived Axiom Schemata and Derived Rules

CLD1 $P \supset P$ (*cf.* e.*g.* 4.11)

CLD2 $(Q \supset R) \supset ((P \supset Q) \supset (P \supset R))$

H.S. (Hypothetical Syllogism) From $P \supset Q$ and $Q \supset R$ infer $P \supset R$

Trans. (Transposition) From $P \supset (Q \supset R)$ infer $Q \supset (P \supset R)$

CLD3 $\sim\sim P \supset P$

CLD4 $P \supset \sim\sim P$

CLD5 $(P \supset Q) \supset (\sim Q \supset \sim P)$

M.T. (*Modus Tollens*) from $P \supset Q$ and $\sim Q$ infer $\sim P$

***E.g.* 4.20** **CLD1** can be used to prove that A v $\sim A$ is a theorem.

First, we rewrite A v $\sim A$ as $\sim A \supset \sim A$ (containing only $\supset$ and $\sim$), then we derive:

1. $\sim A \supset \sim A$ CLD1, $\sim A/P$

A derivation along the lines of *e.g.* 4.11 would also be possible. (p. 228)

Rem. 4.21 Reconsider the derivation of the theorem $\sim F \supset (F \supset G)$ (or $F \vee (F \supset G)$). Except for lines 8 and 9 of the derivation in *e.g.* 4.19 above, the main purpose

is to derive $\sim F \supset (F \supset G)$ line 11

from $\sim F \supset (\sim G \supset \sim F)$and $(\sim G \supset \sim F) \supset (F \supset G)$ lines 8 & 9.

Then line 11 is obtained from line 7 using M.P. twice.

From these formulae, we see that there is a general pattern of inference

that derives a formula of the form $P \supset R$

from the formulae $P \supset Q$ and $Q \supset R$.

This inference pattern can be introduced as a derived rule yielding:

H.S. (Hypothetical Syllogism): From $P \supset Q$ and $Q \supset R$ infer $P \supset R$.

The justification for H.S. can be constructed using the using the derived axiom:

CLD2 $(Q \supset R) \supset ((P \supset Q) \supset (P \supset R))$ (p. 230)

Fronhöfer reproduces the first 7 lines of the derivation below. Compare this to the derivation of $F \vee (F \supset G)$ in *e.g.* 4.19 above.

1.	$((P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))) \supset$ $((Q \supset R) \supset$ $((P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))))$	CL1, $(P \supset (Q \supset R)) \supset$ $((P \supset Q) \supset (P \supset R))/P$, $Q \supset R/Q$
2.	$(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$	CL2, P/P, Q/Q, R/R
3.	$(Q \supset R) \supset$ $((P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R)))$	1,2 M.P.
4.	$((Q \supset R) \supset$ $((P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R)))) \supset$ $(((Q \supset R) \supset (P \supset (Q \supset R))) \supset$ $((Q \supset R) \supset ((P \supset Q) \supset (P \supset R))))$	CL2, $Q \supset R/P$, $P \supset (Q \supset R)/Q$, $(P \supset Q) \supset (P \supset R)/R$
5.	$((Q \supset R) \supset (P \supset (Q \supset R))) \supset$ $((Q \supset R) \supset ((P \supset Q) \supset (P \supset R)))$	3,4 M.P.
6.	$(Q \supset R) \supset (P \supset (Q \supset R))$	CL1, $Q \supset R/P$, P/Q
7.	$(Q \supset R) \supset ((P \supset Q) \supset (P \supset R))$	5,6 M.P. (p. 230)

H.S. (Hypothetical Syllogism): From $P \supset Q$ and $Q \supset R$, infer $P \supset R$.

1.	$P \supset Q$	CPA
2.	$Q \supset R$	CPA
3.	$(Q \supset R) \supset ((P \supset Q) \supset (P \supset R))$	CLD2 P/P, Q/Q, R/R
4.	$(P \supset Q) \supset (P \supset R)$	2,3 M.P.
5.	$P \supset R$	1,4 M.P.

According to the above derivation, if we have $P \supset Q$ and $Q \supset R$, whether or not they are assumptions, we can derive $P \supset R$.

Rem. 4.22 Using H.S. can drastically shorten the derivation of $\sim F \supset (F \supset G)$ in *e.g.* 4.19 above:

1.	$\sim F \supset (\sim G \supset \sim F)$	CL1, $\sim F/P$, $\sim G/Q$	(was line 8 above)
2.	$(\sim G \supset \sim F) \supset (F \supset G)$	CL3, G/P, F/Q	(was line 9 above)
3.	$\sim F \supset (F \supset G)$	1,2 H.S.	(p. 232)

Trans. (Transposition): From $P \supset (Q \supset R)$ infer $Q \supset (P \supset R)$.

1.	$P \supset (Q \supset R)$	CPA	
2.	$((P \supset Q) \supset (P \supset R)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))$	CLD2, Q/P, $P \supset Q/Q$, $P \supset R/R$	
3.	$(((P \supset Q) \supset (P \supset R)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset (P \supset R)))) \supset (((P \supset Q) \supset (P \supset R)) \supset (Q \supset (P \supset Q))) \supset (((P \supset Q) \supset (P \supset R)) \supset (Q \supset (P \supset R)))$	CL2, $(P \supset Q) \supset (P \supset R)/P$, $Q \supset (P \supset Q)/Q$ $Q \supset (P \supset R)/R$	
4.	$((P \supset Q) \supset (P \supset R)) \supset (Q \supset (P \supset Q)) \supset (((P \supset Q) \supset (P \supset R)) \supset (Q \supset (P \supset R)))$	2,3 M.P	
5.	$Q \supset (P \supset Q)$	CL1, Q/P, P/Q	
6.	$(Q \supset (P \supset Q)) \supset (((P \supset Q) \supset (P \supset R)) \supset (Q \supset (P \supset Q)))$	CL1, $Q \supset (P \supset Q)/P$, $(P \supset Q) \supset (P \supset R)/Q$	
7.	$((P \supset Q) \supset (P \supset R)) \supset (Q \supset (P \supset Q))$	5,6 M.P	
8.	$((P \supset Q) \supset (P \supset R)) \supset (Q \supset (P \supset R))$	4,7 M.P.	
9.	$(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$	CL2, P/P, Q/Q, R/R	
10.	$(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$	8,9 H.S.	
11.	$Q \supset (P \supset R)$	1,10 M.P.	(p. 232)

CLD3 $\sim\sim P \supset P$

1.	$\sim\sim P \supset (\sim\sim\sim\sim P \supset \sim\sim P)$	CL1, $\sim\sim P/P$, $\sim\sim\sim\sim P/Q$
2.	$(\sim\sim\sim\sim P \supset \sim\sim P) \supset (\sim P \supset \sim\sim\sim P)$	CL3, $\sim\sim\sim P/P$, $\sim P/Q$
3.	$\sim\sim P \supset (\sim P \supset \sim\sim\sim P)$	1,2 H.S.
4.	$(\sim P \supset \sim\sim\sim P) \supset (\sim\sim P \supset P)$	CL3, P/P, $\sim\sim P/Q$
5.	$\sim\sim P \supset (\sim\sim P \supset P)$	3,4 H.S.
6.	$(\sim\sim P \supset (\sim\sim P \supset P)) \supset$ $((\sim\sim P \supset \sim\sim P) \supset (\sim\sim P \supset P)$	CL2, $\sim\sim P/P$, $\sim\sim P/Q$, P/R
7.	$(\sim\sim P \supset \sim\sim P) \supset (\sim\sim P \supset P)$	5,6 M.P.
8.	$\sim\sim P \supset \sim\sim P$	CLD1, $\sim\sim P/P$
9.	$\sim\sim \mathrm{P} \supset \mathrm{P}$	7,8 M.P.

CLD4 $P \supset \sim\sim P$

1.	$\sim\sim\sim P \supset \sim P$	CLD3, $\sim P/P$	
2.	$(\sim\sim\sim P \supset \sim P) \supset (P \supset \sim\sim P)$	CL3, $\sim\sim P/P$, P/Q	
3.	$\mathrm{P} \supset \sim\sim \mathrm{P}$	1,2 M.P	(p. 232)

CLD5 $(P \supset Q) \supset (\sim Q \supset \sim P)$ (The converse of **CL3**)

1.	$(Q \supset \sim\sim Q) \supset ((P \supset Q) \supset (P \supset \sim\sim Q))$	CLD2, P/P, Q/Q, $\sim\sim Q/R$
2.	$Q \supset \sim\sim Q$	CLD4, Q/P
3.	$(P \supset Q) \supset (P \supset \sim\sim Q)$	1,2 M.P.
4.	$(P \supset \sim\sim Q) \supset ((\sim\sim P \supset P) \supset$ $(\sim\sim P \supset \sim\sim Q))$	CLD2, $\sim\sim P/P$, P/Q, $\sim\sim Q/R$
5.	$(\sim\sim P \supset P) \supset ((P \supset \sim\sim Q) \supset$ $(\sim\sim P \supset \sim\sim Q))$	4 Trans.
6.	$\sim\sim P \supset P$	CLD3, P/P
7.	$(P \supset \sim\sim Q) \supset (\sim\sim P \supset \sim\sim Q)$	5,6 M.P.
8.	$(P \supset Q) \supset (\sim\sim P \supset \sim\sim Q)$	3,7 H.S.
9.	$(\sim\sim P \supset \sim\sim Q) \supset (\sim Q \supset \sim P)$	CL3, $\sim P/P$, $\sim Q/Q$

10. $(P \supset Q) \supset (\sim Q \supset \sim P)$ 8,9 H.S.

M.T. (Modus Tollens): From $P \supset Q$ and $\sim Q$, infer $\sim P$

1.	$P \supset Q$	CPA
2.	$\sim Q$	CPA
3.	$(P \supset Q) \supset (\sim Q \supset \sim P)$	CLD5, P/P, Q/Q
4.	$\sim Q \supset \sim P$	1,3 M.P.
5.	$\sim P$	2,4 M.P.

***E.g.* 4.23** If the economy is sound, then the unemployment rate is low or spending is high. If the unemployment rate is low, then most people are well off. If spending is high, then most people are well off. It's not true that most people are well off. Therefore the economy is not sound.

The above argument can be symbolised as:

$\mathrm{E} \supset (\mathrm{U} \vee \mathrm{S})$
$\mathrm{U} \supset \mathrm{W}$
$\mathrm{S} \supset \mathrm{W}$
$\sim\mathrm{W}$
———
$\therefore \sim\mathrm{E}$

Fronhöfer rewrites the first premise as $\mathrm{E} \supset (\sim\mathrm{U} \supset \mathrm{S})$. This is possible because of the logical equivalence $(p \vee q) \equiv (\sim p \supset q)$; however strictly the justification for this move should have been included in the proof that follows.

1.	$\mathrm{E} \supset (\sim\mathrm{U} \supset \mathrm{S})$	CPA
2.	$\mathrm{U} \supset \mathrm{W}$	CPA
3.	$\mathrm{S} \supset \mathrm{W}$	CPA
4.	$\sim\mathrm{W}$	CPA
5.	$\sim\mathrm{U}$	2,4 M.T.
6.	$\sim\mathrm{S}$	3.4 M.T.
7.	$(\sim\mathrm{U} \supset \mathrm{S}) \supset (\sim\mathrm{S} \supset \sim\sim\mathrm{U})$	CLD5, ~U/P, S/Q
8.	$\sim\mathrm{S} \supset ((\sim\mathrm{U} \supset \mathrm{S}) \supset \sim\sim\mathrm{U})$	7 Trans.
9.	$(\sim\mathrm{U} \supset \mathrm{S}) \supset \sim\sim\mathrm{U}$	6,8 M.P.
10.	$((\sim\mathrm{U} \supset \mathrm{S}) \supset \sim\sim\mathrm{U}) \supset (\sim\sim\sim\mathrm{U} \supset \sim(\sim\mathrm{U} \supset \mathrm{S}))$	CLD5, ~U ⊃ S/P, ~~U/Q
11.	$\sim\sim\sim\mathrm{U} \supset \sim(\sim\mathrm{U} \supset \mathrm{S})$	9,10 M.P.
12.	$\sim\mathrm{U} \supset \sim\sim\sim\mathrm{U}$	CLD4, ~U/P
13.	$\sim\sim\sim\mathrm{U}$	5,12 M.P.
14.	$\sim(\sim\mathrm{U} \supset \mathrm{S})$	11,13 M.P.
15.	$\sim\mathrm{E}$	1,14 M.T.

Rem. 4.24 The derived axiom schema and derived rules are a convenience for constructing derivations. Since the axiom schemata CL1 - CL3 together with the rule M.P. alone form a complete derivation system for classical propositional logic, additional axioms and rules do not increase the power of the system or its soundness, since they are all derivable from within the system that was sound to begin with. (p. 238)

Deduction Theorems

Rem. 4.25 Given any logic with semantic concepts of entailment and tautology and with syntactic/proof-theoretical concepts of derivation and theorems, it is an interesting question as to whether the following theorems obtain in such a logic.

(Syntactic/Proof-Theoretical) Deduction Theorem

- A formula $\mathbf{Q}$ is derivable from a set of formulae $\mathbf{P}_1, \ldots, \mathbf{P}_n$ of formulae iff $\mathbf{P}_n \supset \mathbf{Q}$ is derivable from $\mathbf{P}_1, \ldots, \mathbf{P}_{n-1}$. In the special case of $n = 1$, a formula $\mathbf{Q}$ is derivable from a formula $\mathbf{P}$ iff $\mathbf{P} \supset \mathbf{Q}$ is a theorem.

(Semantic) Deduction Theorem

- A set $\{\mathbf{P}\}$ of formulae entails a formula $\mathbf{Q}$ iff $\mathbf{P} \supset \mathbf{Q}$ is a tautology, *i.e.* $\{\mathbf{P}\} \vDash \mathbf{Q}$ iff $\vDash \mathbf{P} \supset \mathbf{Q}$. (p. 240)

Rem. 4.26 The (Syntactic/Proof-Theoretical) Deduction Theorem is of great practical importance in deriving theorems since it is usually easier to derive $\mathbf{Q}$ from $\mathbf{P}$ than to derive $\mathbf{P} \supset \mathbf{Q}$ directly.

The (Semantic) Deduction Theorem allows entailment to be mapped onto the language being used. (Compare the implication connectives $\supset_{\mathrm{K}}$ and $\rightarrow_{\mathrm{K}}$ (our notation) with the Logic of Paradox.

Both of the above theorems coincide with a sound and complete calculus. (p. 240)

Syntactic Deduction Theorem

Thrm. 4.27 For propositional formulae $\mathbf{P}$ and $\mathbf{Q}$ of classical logic: $\mathbf{Q}$ is derivable from $\mathbf{P}$ in **CLA** if $\mathbf{P} \supset \mathbf{Q}$ is a theorem in **CLA**.

Proof: If $\mathbf{P} \supset \mathbf{Q}$ is a theorem, then there exists a proof/derivation of $\mathbf{P} \supset \mathbf{Q}$ in **CLA**. We can add $\mathbf{P}$ to this proof as the only assumption and then use M.P. to derive $\mathbf{Q}$ from $\mathbf{P}$ and $\mathbf{P} \supset \mathbf{Q}$.

			0	$\mathbf{P}$	CPA
1	$\mathbf{F}_1$	Justification 1	1	$\mathbf{F}_1$	Justification 1

⋮	⋮	⋮	$\supset$	⋮	⋮	⋮
n-1	$\mathbf{F}_{n-1}$	Justification n-1		n-1	$\mathbf{F}_{n-1}$	Justification n-1
n	$\mathbf{P} \supset \mathbf{Q}$	Justification n		n	$\mathbf{P} \supset \mathbf{Q}$	Justification n
				n+1	$\mathbf{Q}$	n,0 M.P

Given a derivation $\mathcal{D}_1$ of $\mathbf{Q}$ from the only assumption $\mathbf{P}$ consisting of the sequence $\mathbf{R}_1$, $\mathbf{R}_2$, … , $\mathbf{R}_{n-1}$, $\mathbf{R}_n$ where $\mathbf{R}_1$ is $\mathbf{P}$ and $\mathbf{R}_n$ is $\mathbf{Q}$, we can produce a new derivation $\mathcal{D}_2$ in which, *inter alia*, each of the formulae $\mathbf{P} \supset \mathbf{R}_1$, $\mathbf{P} \supset \mathbf{R}_2$, … , $\mathbf{P} \supset \mathbf{R}_{n-1}$, $\mathbf{P} \supset \mathbf{R}_n$ occurs as a theorem.

Each of $\mathbf{R}_1$, $\mathbf{R}_2$, … , $\mathbf{R}_{n-1}$, $\mathbf{R}_n$ will either be an assumption, an instance of an axiom schema, or will follow by M.P. from previous formulae in the derivation.

- In the case that $\mathbf{R}_i$ is our assumption ($\mathbf{R}_i = \mathbf{R}_1 = \mathbf{P}$), we derive $\mathbf{R}_i \supset \mathbf{R}_i$ as we did for $A \supset A$ in *e.g.* 4.11 above.

- In the case that $\mathbf{R}_i$ is an instance of an axiom schema from step i of $\mathcal{D}_1$, we may add the following lines to derive $\mathbf{P} \supset \mathbf{R}_i$ in the new derivation, thus:

m	$\mathbf{R}_i$	by the relevant axiom schema
m+1	$\mathbf{R}_i \supset (\mathbf{P} \supset \mathbf{R}_i)$	CL1, $\mathbf{R}_i/\mathbf{P}$, $\mathbf{P}/\mathbf{Q}$
m+2	$\mathbf{P} \supset \mathbf{R}_i$	m+1,m+2 M.P.

Note that m is the step in $\mathcal{D}_2$ to which step i of $\mathcal{D}_1$ has been shifted.

- In the third case, $\mathbf{R}_i$ follows by M.P. from earlier formulae $\mathbf{R}_k$ and $\mathbf{R}_k \supset \mathbf{R}_i$ in the sequence $\mathbf{R}_1$, $\mathbf{R}_2$, … , $\mathbf{R}_{n-1}$, $\mathbf{R}_n$, *i.e.* where $\mathbf{R}_k$ and $\mathbf{R}_k \supset \mathbf{R}_i$ are among $\mathbf{R}_1$, $\mathbf{R}_2$, … , $\mathbf{R}_{i-1}$ in $\mathcal{D}_1$. If so, we already have $\mathbf{P} \supset \mathbf{R}_k$ and $\mathbf{P} \supset (\mathbf{R}_k \supset \mathbf{R}_i)$ in the new derivation $\mathcal{D}_2$, say on lines m and n (by induction hypothesis). Three more lines are required to derive $\mathbf{P} \supset \mathbf{R}_i$:

m	$\mathbf{P} \supset \mathbf{R}_k$	
⋮		
n	$\mathbf{P} \supset (\mathbf{R}_k \supset \mathbf{R}_i)$	
⋮		
o	$(\mathbf{P} \supset (\mathbf{R}_k \supset \mathbf{R}_i)) \supset ((\mathbf{P} \supset \mathbf{R}_k) \supset (\mathbf{P} \supset \mathbf{R}_i))$	CL2, $\mathbf{P}/\mathbf{P}$, $\mathbf{R}_k/\mathbf{Q}$, $\mathbf{R}_i/\mathbf{R}$
o+1	$(\mathbf{P} \supset \mathbf{R}_k) \supset (\mathbf{P} \supset \mathbf{R}_i)$	o,n M.P.
o+2	$\mathbf{P} \supset \mathbf{R}_i$	o+1,m M.P.

Note that o is the step in $\mathcal{D}_2$ to which i of $\mathcal{D}_1$ has been shifted. (p. 242 - 244)

***E.g.* 4.28** Using the above method, construct a derivation establishing that $\sim A \supset (A \supset B)$ is a theorem.

In example 4.8, reproduced below, Fronhöfer showed that $A \supset B$ is derivable from $\sim A$.

1.	$\sim A$	CPA
2.	$\sim A \supset (\sim B \supset \sim A)$	CL1, $\sim A/P$, $\sim B/Q$
3.	$\sim B \supset \sim A$	2,1 M.P.
4.	$(\sim B \supset \sim A) \supset (A \supset B)$	CL3, B/P, A/Q
5.	$A \supset B$	4,3 M.P.

We can now construct a derivation establishing that $\sim A \supset (A \supset B)$ is a theorem. The derivation below is set out so that lines containing conditionals whose consequents are formulae from the earlier derivation are marked with a red numeral to the right for the line number of that derivation. According to Fronhöfer, shorter derivations are also possible. *E.g.* there is no need to derive the formula on line 11 since it already appears on line 6. (p. 244)

1.	$\sim A \supset ((\sim A \supset \sim A) \supset \sim A)$	CL1, $\sim A/P$, $\sim A \supset \sim A/Q$
2.	$(\sim A \supset ((\sim A \supset \sim A) \supset \sim A)) \supset$ $((\sim A \supset (\sim A \supset \sim A)) \supset (\sim A \supset \sim A))$	CL2, $\sim A/P$, $\sim A \supset \sim A/Q$ $\sim A/R$
3.	$(\sim A \supset (\sim A \supset \sim A)) \supset (\sim A \supset \sim A)$	2,1 M.P.
4.	$\sim A \supset (\sim A \supset \sim A)$	CL1, $\sim A/P$, A/Q
5.	$\sim A \supset \sim A$	1. 3,4 M.P.
6.	$\sim A \supset (\sim B \supset \sim A)$	CL1, $\sim A/P$, $\sim B/Q$
7.	$(\sim A \supset (\sim B \supset \sim A)) \supset$ $(\sim A \supset (\sim A \supset (\sim B \supset \sim A)))$	CL1, $\sim A \supset (\sim B \supset \sim A)/P$ $\sim A/Q$
8.	$\sim A \supset (\sim A \supset (\sim B \supset \sim A))$	2. 7,6 M.P.
9.	$(\sim A \supset (\sim A \supset (\sim B \supset \sim A))) \supset$ $((\sim A \supset \sim A) \supset (\sim A \supset (\sim B \supset \sim A)))$	CL2, $\sim A/P$, $\sim A/Q$ $\sim A \supset \sim B/R$
10.	$(\sim A \supset \sim A) \supset (\sim A \supset (\sim B \supset \sim A))$	9,8 M.P.
11.	$\sim A \supset (\sim B \supset \sim A)$	3. 10,5 M.P.
12.	$(\sim B \supset \sim A) \supset (A \supset B)$	CL3, B/P, A/Q
13.	$((\sim B \supset \sim A) \supset (A \supset B)) \supset$ $(\sim A \supset ((\sim B \supset \sim A) \supset (A \supset B)))$	CL1, $\sim A/Q$ $(\sim B \supset \sim A) \supset (A \supset B)/P$

14. $\sim A \supset ((\sim B \supset \sim A) \supset (A \supset B))$ 4. 13,12 M.P.

15. $(\sim A \supset ((\sim B \supset \sim A) \supset (A \supset B))) \supset$ CL2, $\sim A/P$, $\sim B \supset \sim A/Q$
$((\sim A \supset (\sim B \supset \sim A)) \supset (\sim A \supset (A \supset B)))$ $A \supset B/R$

16. $(\sim A \supset (\sim B \supset \sim A)) \supset (\sim A \supset (A \supset B))$ 15,14 M.P.

17. $\sim A \supset (A \supset B)$ 5. 16,11 M.P.

(P. 246)

An Axiomatic System for Łukasiewicz's 3-Valued Logic

Rem. 4.29 Both Kleene's and Bochvar's connectives can be defined using those of Łukasiewicz; therefore we can represent inferences for the first two systems within Ł$_3$ axiomatic systems. Note that, Fronhöfer is working towards fuzzy logic, in which the bulk of formal work is based upon Łukasiewicz's infinite-valued generalization of his 3-valued system.

Proof of Soundness and Completeness of the Axiomatic System for Ł$_3$

Defn. 4.30 The Ł$_3$**A** axiomatic system of Wajsberg, 1931

Ł$_3$**1** $(P \supset (Q \supset P))$

Ł$_3$**2** $(P \supset Q) \supset ((Q \supset R) \supset (P \supset R))$

Ł$_3$**3** $(\sim P \supset \sim Q) \supset (Q \supset P)$

Ł$_3$**4** $((P \supset \sim P) \supset P) \supset P$

M.P. From P and $P \supset Q$ infer Q (p. 250)

Rem. 4.31 Using the definitions of the connectives v, • and ≡ all formulae in Ł$_3$ can be expressed using ~ and ⊃ that appear in Wajsberg's axiomatic system above.

Axiom schemata Ł$_3$**1** and Ł$_3$**3** are identical to **CL1** and **CL3** respectively for classical logic.

The axiom schema **CL2** $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$ is not a axiom of Ł$_3$**A** and is not derivable within Ł$_3$**A** since it is not a tautology in Ł$_3$. On the other hand schemata Ł$_3$**2** and Ł$_3$**4** are derivable in **CLA** since they are classical tautologies and the classical logic system is complete. Therefore, all the axioms of Ł$_3$**1** to Ł$_3$**4** are classical tautologies.

Any derivation in **CLA** that doesn't involve **CL2** will be a derivation in Ł$_3$**A** and any axiom that is derivable in **CLA** without using **CL2** will be a derivation in Ł$_3$**A**.

Rem. 4.32 Consider next **CLD1**'s $P \supset P$. Is there a different derivation? $Ł_3\mathbf{4}$'s $((P \supset \sim P) \supset P) \supset P$ looks like a possible candidate. Recall that P v Q may be defined as $(P \supset Q) \supset Q$ in $Ł_3$. Therefore the axiom schema of $Ł_3\mathbf{4}$ may be rewritten as $(P \supset \sim P)$ v P, or as $(\sim_{BE}P \supset P) \supset P$).

According to Fronhöfer, this is closely related to the Law of the Excluded Middle. However, the Law of the Excluded Middle is not a tautology in $Ł_3$ but $(P \supset \sim P)$ v P is a tautology in $Ł_3$.

- If P is T then $(P \supset \sim P)$ v P is T because P is the right disjunct, and if
- if P is ⊙ or F, then the left conjunct is T, hence $(P \supset \sim P)$ v P is T.

Rem. 4.33 $Ł_3\mathbf{4}$'s $((P \supset Q) \supset P) \supset P$ is a classical tautology, but not a tautology in $Ł_3$. If Q is replaced by $\sim P$ below the contingency under the major connective is eliminated.

((P	⊃	Q)	⊃	P)	⊃	P
T	T	T	T	T	T	T
T	⊙	⊙	T	T	T	T
T	F	F	T	T	T	T
⊙	T	T	⊙	⊙	T	⊙
⊙	T	⊙	⊙	⊙	T	⊙
⊙	⊙	F	T	⊙	⊙	⊙
F	T	T	F	F	T	F
F	T	⊙	F	F	T	F
F	T	F	F	F	T	F

Derived Axioms and Rules I

$Ł_3$**D1** $\sim P \supset (P \supset Q)$

$Ł_3$**D2** $\sim\sim P \supset P$

H.S. (Hypothetical Syllogism) From $P \supset Q$ and $Q \supset R$, infer $P \supset R$.

$Ł_3$**D3** $P \supset \sim\sim P$

$Ł_3$**D4** $P \supset P$ (new proof required for $Ł_3$)

$Ł_3$**D5** $((P \supset P) \supset Q) \supset Q$

$Ł_3$**D6** $P \supset ((P \supset Q) \supset Q)$

$Ł_3$**D7** $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$

Con. (Contraposition) From $\sim P \supset \sim Q$ infer $Q \supset P$.

LSimp. (Left Conjunct Simplification) From $P \bullet Q$ infer P.

RSimp. (Right Conjunct Simplification) From $P \bullet Q$ infer Q.

Sub. (Substitution) From $P \supset Q$, $Q \supset P$ and a formula R that contains P as a sub-formula, infer any formula R^* that is the result of replacing one or more occurrences of P in R with Q.

M.T. (Modus Tollens) From $Q \supset P$ and $\sim P$ derive $\sim Q$.

D.N. (Double Negation) From any formula R that contains P as a constituent, infer any formula R^* that is the result of replacing one or more occurrences of P in R with $\sim\sim P$ and *vice versa*. (p. 254)

Derived Axioms and Rules II

Tran. (Transposition) For any formula R that contains $P \supset (Q \supset S)$ as a sub-formula, infer any formula R^* that is the result of replacing one or more occurrences of $P \supset (Q \supset S)$ in R with $Q \supset (P \supset S)$.

GCon. (Generalised Contraposition) For any formula R that contains $P \supset Q$ as a sub-formula, infer any formula R^* that is the result of replacing one or more occurrences of $P \supset Q$ in R with $\sim Q \supset \sim P$ and *vice versa*.

Ł$_3$D8 $\sim(P \supset Q) \supset P$

GHS (Generalised Hypothetical Syllogism) From $(P_1 \supset (P_2 \supset \cdots \supset (P_{n-1} \supset P_n) \ldots)$ and $P_n \supset Q$, infer $(P_1 \supset (P_2 \supset \cdots \supset P_{n-1} \supset Q) \ldots)$.

GMP (Generalised Modus Ponens) From $(P_1 \supset (P_2 \supset \cdots \supset (P_{n-1} \supset P_n) \ldots)$ and one of the antecedents P_i, $1 \leq i \leq n-1$, infer the conditional that results from deleting P_i, the conditional hook following P_i, and associated parentheses.

MCD (Modified Constructive Dilemma) From $P \supset Q$ and $(P \supset \sim P) \supset Q$, infer Q.

DE (Disjunction Elimination) From $P \vee Q$, $P \supset R$ and $Q \supset R$, infer R.

DC (Disjunctive Consequence) From $P \supset R$ and $Q \supset R$, infer $(P \vee Q) \supset R$.

Ł$_3$D9 $(P \vee Q) \supset (Q \vee P)$

Ł$_3$D10 $(P \supset Q) \vee (Q \supset P)$

Ł$_3$D11 $(P \supset (P \supset (P \supset Q))) \supset (P \supset (P \supset Q))$

C.I. (Conjunction Introduction) From P and Q, infer $P \bullet Q$. (p. 254)

Fronhöfer's proofs follow:

Ł$_3$D1 $\sim P \supset (P \supset Q)$

1. $\sim P \supset (\sim Q \supset \sim P)$ — Ł$_3$1, $\sim P/P$, $\sim Q/Q$

2. $(\sim Q \supset \sim P) \supset (P \supset Q)$ — Ł$_3$3, Q/P, P/Q

3.	$(\sim P \supset (\sim Q \supset \sim P)) \supset$ $(((\sim Q \supset \sim P) \supset (P \supset Q)) \supset$ $(\sim P \supset (P \supset Q)))$	Ł$_3$2, $\sim P/P$, $\sim Q \supset \sim P/Q$, $P \supset Q/R$	
4.	$((\sim Q \supset \sim P) \supset (P \supset Q)) \supset$ $(\sim P \supset (P \supset Q))$	3,1 M.P.	
5.	$\sim P \supset (P \supset Q))$	4,2 M.P.	(p. 256)

H.S. (Hypothetical Syllogism) From $P \supset Q$ and $Q \supset R$, infer $P \supset R$.

1.	$P \supset Q$	CPA	
2.	$Q \supset R$	CPA	
3.	$(P \supset Q) \supset ((Q \supset R) \supset (P \supset R))$	Ł$_3$2	
4.	$(Q \supset R) \supset (P \supset R)$	3,1 M.P.	
5.	$P \supset R$	4,2 M.P.	(p. 256)

Ł$_3$D2 $\sim\sim P \supset P$

1.	$\sim\sim P \supset (\sim P \supset \sim(P \supset \sim P))$	Ł$_3$D1, $\sim P/P$, $\sim(P \supset \sim P)/Q$
2.	$(\sim P \supset \sim(P \supset \sim P)) \supset$ $((P \supset \sim P) \supset P)$	Ł$_3$3, P/P, $P \supset \sim P/Q$
3.	$\sim\sim P \supset ((P \supset \sim P) \supset P)$	1,2 H.S.
4.	$((P \supset \sim P) \supset P) \supset P$	Ł$_3$4, P/P
5.	$\sim\sim P \supset P$	3,4 H.S.

Ł$_3$D3 $P \supset \sim\sim P$

1.	$\sim\sim\sim P \supset \sim P$	Ł$_3$D2, $\sim P/P$	
2.	$(\sim\sim\sim P \supset \sim P) \supset (P \supset \sim\sim P)$	Ł$_3$3, $\sim\sim P/P$, P/Q	
3.	$P \supset \sim\sim P$	2,1 M.P.	(p. 256)

Ł$_3$D4 $P \supset P$ (= **CLD1**)

1.	$P \supset \sim\sim P$	Ł$_3$D3, P/P

2.	$\sim\sim P \supset P$	$Ł_3$D2, P/P	
3.	$P \supset P$	1,2 H.S.	(p. 258)

$Ł_3$D5 $((P \supset P) \supset Q) \supset Q$

1.	$(P \supset P) \supset ((Q \supset \sim Q) \supset (P \supset P))$	$Ł_3$1, $P \supset P$/P, $Q \supset \sim Q/Q$	
2.	$P \supset P$	$Ł_3$D4, P/P	
3.	$(Q \supset \sim Q) \supset (P \supset P)$	1,2 M.P	
4.	$((Q \supset \sim Q) \supset (P \supset P)) \supset$ $(((P \supset P) \supset Q) \supset (Q \supset \sim Q) \supset Q))$	$Ł_3$2, $Q \supset \sim Q/P$, $P \supset P/Q$, Q/R	
5.	$((P \supset P) \supset Q) \supset ((Q \supset \sim Q) \supset Q)$	4,3 M.P.	
6.	$((Q \supset \sim Q) \supset Q) \supset Q$	$Ł_3$4, Q/P	
7.	$((P \supset P) \supset Q) \supset Q$	5,6 H.S.	(p. 258)

Note: The following formula is equivalent to $P \supset (P \vee Q)$ when rewritten with $Ł_3$ disjunction.

$Ł_3$D6 $P \supset ((P \supset Q) \supset Q)$

1.	$P \supset ((P \supset P) \supset P)$	$Ł_3$1, P/P, $P \supset P/Q$
2.	$((P \supset P) \supset P) \supset$ $((P \supset Q) \supset ((P \supset P) \supset Q))$	$Ł_3$2, $P \supset P/P$, P/Q, Q/R
3.	$P \supset ((P \supset Q) \supset ((P \supset P) \supset Q))$	1,2 H.S.
4.	$((P \supset Q) \supset ((P \supset P) \supset Q)) \supset$ $((((P \supset P) \supset Q) \supset Q) \supset ((P \supset Q) \supset Q))$	$Ł_3$2, $P \supset Q/P$, $(P \supset P) \supset Q/Q$, Q/R
5.	$P \supset ((((P \supset P) \supset Q) \supset Q) \supset$ $((P \supset Q) \supset Q))$	3,4 H.S.
6.	$((P \supset P) \supset Q) \supset Q$	$Ł_3$D5, P/P, Q/Q
7.	$(((P \supset P) \supset Q) \supset Q) \supset$ $((P \supset P) \supset (((P \supset P) \supset Q) \supset Q))$	$Ł_3$1, $((P \supset P) \supset Q) \supset Q/P$ $P \supset P/Q$
8.	$(P \supset P) \supset (((P \supset P) \supset Q) \supset Q)$	6,7 M.P.
9.	$((P \supset P) \supset (((P \supset P) \supset Q) \supset Q)) \supset$ $(((((P \supset P) \supset Q) \supset Q) \supset ((P \supset Q) \supset Q)) \supset$	$Ł_3$1, $P \supset P/P$, $((P \supset P) \supset Q) \supset Q/Q$,

	$((P \supset P) \supset ((P \supset Q) \supset Q)))$	$(P \supset Q) \supset Q/R$
10.	$((((P \supset P) \supset Q) \supset Q) \supset ((P \supset Q) \supset Q)) \supset$ $((P \supset P) \supset ((P \supset Q) \supset Q))$	8,9 M.P.
11.	$P \supset ((P \supset P) \supset ((P \supset Q) \supset Q))$	5,10 H.S.
12.	$((P \supset P) \supset ((P \supset Q) \supset Q)) \supset ((P \supset Q) \supset Q)$	Ł$_3$D5 P/P, $(P \supset Q) \supset Q/Q$
13.	$P \supset ((P \supset Q) \supset Q)$	11,12 H.S. (p. 258 - 260)

Rem. 4.34 $P \supset (Q \vee P)$ (which is equivalent to $P \supset (P \vee Q)$ in classical logic) is the instance of $P \supset ((Q \supset P) \supset P)$ of Ł$_3$1 when rewritten with disjunction. (p. 260)

Ł$_3$D7 $(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$

1.	$(P \supset (Q \supset R)) \supset$ $(((Q \supset R) \supset R) \supset (P \supset R))$	Ł$_3$2, P/P, $Q \supset R/Q$, R/R
2.	$Q \supset ((Q \supset R) \supset R)$	Ł$_3$D6, Q/Q, R/R
3.	$(Q \supset ((Q \supset R) \supset R)) \supset$ $((((Q \supset R) \supset R) \supset (P \supset R)) \supset$ $(Q \supset (P \supset R)))$	Ł$_3$2, Q/P, $(Q \supset R) \supset R/Q$ $P \supset R/R$
4.	$(((Q \supset R) \supset R) \supset (P \supset R)) \supset$ $(Q \supset (P \supset R)))$	3,2 M.P.
5.	$(P \supset (Q \supset R)) \supset (Q \supset (P \supset R))$	1,4 H.S. (p, 260)

Rem. 4.35 An implication formula $P \supset Q$ allows us to construct an instance, $F \supset G$ and thence deduce G from F by means of M.P. For some formulae $P \supset Q$, this usage is so common that it is convenient to introduce a respective derived rule. (p. 262)

Con. (Contraposition) From $\sim P \supset \sim Q$ infer $Q \supset P$.

1.	$\sim P \supset \sim Q$	CPA
2.	$(\sim P \supset \sim Q) \supset (Q \supset P)$	Ł$_3$3, P/P, Q/Q
3.	$Q \supset P$	2,1 M.P (p. 262)

LSimp. (Left Conjunct Simplification) From $P \bullet Q$ infer P.

1.	$\sim((\sim P \supset \sim Q) \supset \sim Q)$	CPA (rewritten from $P \bullet Q$)
2.	$\sim P \supset ((\sim P \supset \sim Q) \supset \sim Q)$	Ł$_3$D6, $\sim P/P$, $\sim Q/Q$
3.	$((\sim P \supset \sim Q) \supset \sim Q) \supset$ $\sim\sim((\sim P \supset \sim Q) \supset \sim Q)$	Ł$_3$D3, $(\sim P \supset \sim Q) \supset \sim Q/P$
4.	$\sim P \supset \sim\sim((\sim P \supset \sim Q) \supset \sim Q)$	2,3 H.S.
5.	$\sim((\sim P \supset \sim Q) \supset \sim Q) \supset P$	4 Con.
6.	P	1,5 M.P

RSimp. (Right Conjunct Simplification) From $P \bullet Q$ infer Q.

1.	$\sim((\sim P \supset \sim Q) \supset \sim Q)$	CPA (rewritten from $P \bullet Q$)
2.	$\sim Q \supset ((\sim P \supset \sim Q) \supset \sim Q)$	Ł$_3$1, $\sim Q/P$, $\sim P \supset \sim Q/Q$
3.	$((\sim P \supset \sim Q) \supset \sim Q) \supset$ $\sim\sim((\sim P \supset \sim Q) \supset \sim Q)$	Ł$_3$D3, $(\sim P \supset \sim Q) \supset \sim Q/P$
4.	$\sim Q \supset \sim\sim((\sim P \supset \sim Q) \supset \sim Q)$	2,3 H.S.
5.	$\sim((\sim P \supset \sim Q) \supset \sim Q) \supset Q$	4 Con.
6.	Q	1,5 M.P. (p. 262)

Rem. 4.36 Valid inferences using Kleene's and Bochvar's (internal and external) connectives have corresponding derivations in Ł$_3$A provided that we use the Ł$_3$ definitions for rewriting formulae containing such connectives.

***E.g.* 4.37** $P \supset_{\mathrm{BE}} P$ is a tautology, therefore we would expect $\sim(P \supset \sim P) \supset \sim(P \supset \sim P)$, which expresses $P \supset_{\mathrm{BE}} P$ in Ł$_3$ as a theorem of Ł$_3$A. However this is an instance of Ł$_3$D4 with $\sim(P \supset \sim P)/P$. (p. 264)

***E.g.* 4.38** P
$P \supset_{\mathrm{BE}} Q$
$\therefore Q$ of Bochvar's external system $\mathbf{B_3^E}$ is also valid in Ł$_3$A. See below:

1. P CPA

2. $\sim(P \supset \sim P) \supset \sim(Q \supset \sim Q)$ — CPA (rewritten from $P \supset_{\mathrm{BE}} Q$)

3. $(Q \supset \sim Q) \supset (P \supset \sim P)$ — 2 Con.

4. $((Q \supset \sim Q) \supset (P \supset \sim P)) \supset (P \supset ((Q \supset \sim Q) \supset \sim P)))$ — Ł$_3$D7, $Q \supset \sim Q/P$, P/Q, $\sim P/R$

5. $P \supset ((Q \supset \sim Q) \supset \sim P)$ — 4,3 M.P.

6. $(Q \supset \sim Q) \supset \sim P$ — 5,1 M.P.

7. $\sim\sim(Q \supset \sim Q) \supset (Q \supset \sim Q)$ — Ł$_3$D2, $Q \supset \sim Q/P$

8. $\sim\sim(Q \supset \sim Q) \supset \sim P$ — 7,6 H.S.

9. $P \supset \sim(Q \supset \sim Q)$ — 8 Con.

10. $\sim(\mathrm{Q} \supset \sim\mathrm{Q})$ — 9,1 M.P.

11. $\sim Q \supset (Q \supset \sim Q)$ — Ł$_3$1, $\sim Q/P$, Q/Q

12. $(Q \supset \sim Q) \supset \sim\sim(Q \supset \sim Q)$ — Ł$_3$D3, $Q \supset \sim Q/P$

13. $\sim Q \supset \sim\sim(Q \supset \sim Q)$ — 11,12 H.S.

14. $\sim(Q \supset \sim Q) \supset Q$ — 13 Con.

15. Q — 14,10 M.P. (p. 264)

***E.g.* 4.39** P

$P \supset_{\mathrm{K}} Q$

$\therefore Q$ is valid in $\mathbf{K}_3^{\mathbf{S}}$.

$P \supset_{\mathrm{K}} Q$ is equivalent to $\sim P \vee Q$ in Ł$_3$, which is equivalent to $(\sim P \supset Q) \supset Q$. See the following derivation which establishes the validity of the argument above:

1. P — CPA

2. $(\sim P \supset Q) \supset Q$ — CPA

3. $P \supset (\sim Q \supset P)$ — Ł$_3$1, P/P, $\sim Q/Q$

4. $\sim Q \supset P$ — 3,1 M.P.

5. $P \supset {\sim}{\sim}P$ — Ł$_3$D3, P/P

6. ${\sim}Q \supset {\sim}{\sim}P$ — 4,5 H.S.

7. ${\sim}P \supset Q$ — 6 Con.

8. Q — 2,7 M.P

This derivation justifies Disjunctive Syllogism in Ł$_3$. (p. 266)

***E.g.* 4.40** P
$P \supset_{\mathrm{BI}} Q$
$\therefore Q$ is valid in $\mathbf{B}_3^{\mathbf{I}}$.

The second premise above is equivalent to $({\sim}P \text{ v } Q) \bullet ((P \text{ v } {\sim}P) \bullet (Q \text{ v } {\sim}Q))$ in Ł$_3$. The derivation of the argument is as follows:

1. P — CPA

2. $({\sim}P \text{ v } Q) \bullet ((P \text{ v } {\sim}P) \bullet (Q \text{ v } {\sim}Q))$ — CPA

3. ${\sim}P \text{ v } Q$ — 2 LSimp.

4. …

By substituting $({\sim}P \supset Q) \supset Q$ for ${\sim}P \text{ v } Q$, the rest of the proof is identical to that of the $\mathbf{K}_3^{\mathbf{S}}$ proof in *e.g.* 4.39 above. (p. 266)

Sub. (Substitution) From $P \supset Q$, $Q \supset P$ and a formula R that contains P as a sub-formula, infer any formula R^* that is the result of replacing one or more occurrences of P in R with Q.

Fronhöfer's Proof Strategy:

- <u>Structural Induction:</u> If we can derive reciprocal formulae $P \supset Q$ and $Q \supset P$, then given any formula R that contains P we can derive both $R \supset R^+$ and $R^+ \supset R$, where R^+ is identical to R except for one occurrence of P being replaced by Q. It follows then that if we can derive R, we can also derive R^+ by Modus Ponens and *vice versa*.

- <u>Induction on number of replacements:</u> We can then replace more than one occurrence of P in R with Q to obtain any R^* by replacing one occurrence at a time. (p. 268)

***E.g.* 4.41** Here Fronhöfer shows how to derive both $R \supset R^+$ and $R^+ \supset R$ by deriving larger and larger conditionals reflecting the way that R has been constructed from P and hence the way that R^+ is constructed from Q.

Let R be the formula:

$(\sim(A \supset P) \supset (A \supset B)) \supset C$

where there is an occurrence of P which we want to replace by Q. Then R must be constructed from P stepwise by combining P with other formulae (left list below) and R^+ must be constructed from Q stepwise as follows (right list below):

P	Q
$A \supset P$	$A \supset Q$
$\sim(A \supset P)$	$\sim(A \supset Q)$
$\sim(A \supset P) \supset (A \supset B)$	$\sim(A \supset Q) \supset (A \supset B)$
$(\sim(A \supset P) \supset (A \supset B)) \supset C$	$(\sim(A \supset Q) \supset (A \supset B)) \supset C$

Next Fronhöfer shows how to derive the reciprocal conditionals that pair off the formulae from each row of the two lists above *i.e.*

$P \supset Q$ and
$Q \supset P$

$(A \supset P) \supset (A \supset Q)$ and
$(A \supset Q) \supset (A \supset P)$

$\sim(A \supset P) \supset \sim(A \supset Q)$ and
$\sim(A \supset Q) \supset \sim(A \supset P)$

$(\sim(A \supset P) \supset (A \supset B)) \supset (\sim(A \supset Q) \supset (A \supset B))$ and
$(\sim(A \supset Q) \supset (A \supset B)) \supset (\sim(A \supset P) \supset (A \supset B))$

$((\sim(A \supset P) \supset (A \supset B)) \supset C) \supset ((\sim(A \supset Q) \supset (A \supset B)) \supset C)$ and
$((\sim(A \supset Q) \supset (A \supset B)) \supset C) \supset ((\sim(A \supset P) \supset (A \supset B)) \supset C)$

where the last pair of conditionals are

$R \supset R^+$ and
$R^+ \supset R$

as per Fronhöfer's example. (p. 270)

Proof:

- The derivability of $P \supset Q$ and $Q \supset P$ is given in the statement of the rule Sub. (Substitution).

- For each pair of conditionals $S_1 \supset S_2$ and $S_2 \supset S_1$ in the list of paired conditionals, the following pair $T_1 \supset T_2$ and $T_2 \supset T_1$ have S_1 as an immediate component of T_1, and T_2 results from replacing one occurrence of S_1 in T_1 with S_2.

- Given this general pattern for constructing the target formulae, we need only show that, given any formulae $S_1 \supset S_2$ and $S_2 \supset S_1$, there is a way to derive $T_1 \supset T_2$ and $T_2 \supset T_1$, where S_1 is an immediate component of T_1, and T_2 is the result of replacing one occurrence of S_1 in T_1 with S_2.

- There are three possible cases reflecting the structure of T_1 (and hence also T_2).

<u>Case 1:</u> T_1 is $S_1 \supset U$ for some formula U, and T_2 is $S_2 \supset U$. Given $S_1 \supset S_2$ we can derive $(T_2 \supset T_1)$ *i.e.* $\equiv (S_2 \supset U) \supset (S_1 \supset U)$ as follows:

n	$S_1 \supset S_2$	given
$n+1$	$(S_1 \supset S_2) \supset ((S_2 \supset U) \supset (S_1 \supset U))$	Ł$_3$2, S_1/P, S_2/Q, U/R
$n+2$	$(S_2 \supset U) \supset (S_1 \supset U)$	n, $n+1$ M.P.

$T_1 \supset T_2$, which is $(S_1 \supset U) \supset (S_2 \supset U)$, is similarly derived from $S_2 \supset S_1$.
(p. 270)

<u>Case 2:</u> T_1 is $U \supset S_1$ for some formula U, and T_2 is $U \supset S_2$.

n	$S_1 \supset S_2$	given
$n+1$	$(U \supset S_1) \supset ((S_1 \supset S_2) \supset (U \supset S_2))$	Ł$_3$2, U/P, S_1/Q, S_2/R
$n+2$	$((U \supset S_1) \supset ((S_1 \supset S_2) \supset (U \supset S_2))) \supset$ $((S_1 \supset S_2) \supset ((U \supset S_1) \supset (U \supset S_2))$	Ł$_3$D7, $U \supset S_1/P$, $S_1 \supset S_2$/Q, $U \supset S_2/R$
$n+3$	$(S_1 \supset S_2) \supset ((U \supset S_1) \supset (U \supset S_2))$	$n+1$, $n+2$ M.P.
$n+4$	$((U \supset S_1) \supset (U \supset S_2))$ *i.e.* $\equiv (T_1 \supset T_2)$	n, $n+3$ M.P.

$T_2 \supset T_1$, which is $(U \supset S_2) \supset (U \supset S_1)$, is similarly derived from $S_2 \supset S_1$.
(p. 272)

<u>Case 3:</u> T_1 is $\sim S_1$ and T_2 is $\sim S_2$.

n	$S_1 \supset S_2$	given
$n+1$	$\sim\sim S_1 \supset S_1$	Ł$_3$D2, S_1/P
$n+2$	$\sim\sim S_1 \supset S_2$	n, $n+1$ H.S.
$n+3$	$S_2 \supset \sim\sim S_2$	Ł$_3$D3, S_2/P
$n+4$	$\sim\sim S_1 \supset \sim\sim S_2$	$n+2$, $n+3$ H.S.

$n+5$ $(\sim S_2 \supset \sim S_1)$ *i.e.* $\equiv (T_2 \supset T_1)$ $n+4$ Con.

$T_1 \supset T_2$, which is $\sim S_1 \supset \sim S_2$, is similarly derived from $S_2 \supset S_1$. (p. 272)

***E.g.* 4.42** We can use Sub. to derive $((\sim\sim Q \supset \sim Q) \supset \sim\sim Q) \supset Q$ as follows:

1.	$((Q \supset \sim Q) \supset Q) \supset Q$	Ł$_3$4, Q/P
2.	$Q \supset \sim\sim Q$	Ł$_3$D3, Q/P
3.	$\sim\sim Q \supset Q$	Ł$_3$D2, Q/P
4.	$((\sim\sim Q \supset \sim Q) \supset \sim\sim Q) \supset Q$	1,2,3 Sub.

On line 4, two occurrences of Q on line 1 were replaced with $\sim\sim Q$. (p. 272)

M.T. (Modus Tollens) From $Q \supset P$ and $\sim P$ derive $\sim Q$.

m	$\sim P$	given	
n	$Q \supset P$	given	
$n+1$	$\sim\sim Q \supset Q$	Ł$_3$D2, Q/P	
$n+2$	$\sim\sim Q \supset P$	n, $n+1$ H.S.	
$n+3$	$P \supset \sim\sim P$	Ł$_3$D3 P/P	
$n+4$	$\sim\sim Q \supset \sim\sim P$	$n+2$, $n+3$ H.S.	
$n+5$	$\sim P \supset \sim Q$	$n+4$, Con.	
$n+6$	$\sim Q$	$n+5$, m M.P.	(p. 274)

D.N. (Double Negation) From any formula R that contains P as a constituent, infer any formula R^* that is the result of replacing one or more occurrences of P in R with $\sim\sim P$ and *vice versa*.

Tran. (Transposition) For any formula R that contains $P \supset (Q \supset S)$ as a sub-formula, infer any formula R^* that is the result of replacing one or more occurrences of $P \supset (Q \supset S)$ in R with $Q \supset (P \supset S)$.

GCon. (Generalised Contraposition) For any formula R that contains $P \supset Q$ as a sub-formula, infer any formula R^* that is the result of replacing one or more occurrences of $P \supset Q$ in R with $\sim Q \supset \sim P$ and *vice versa*.

Proofs:

- **D.N.** follows from Sub. and Ł$_3$D2 as well as Ł$_3$D3.

- **Tran.** follows from Sub. and Ł$_3$D7.

- **GCon.** follows from Sub., Ł$_3$3 and the fact that every formula of the form:

  $(P \supset Q) \supset (\sim Q \supset \sim P)$

  is a theorem of Ł$_3$A, thus:

1.	$(\sim\sim P \supset \sim\sim Q) \supset (\sim Q \supset \sim P)$	Ł$_3$3, $\sim P/P$, $\sim Q/Q$	
2.	$(P \supset Q) \supset (\sim Q \supset \sim P)$	1 D.N. (twice)	(p. 274)

Ł$_3$D8 $\sim(P \supset Q) \supset P$

1.	$\sim P \supset (P \supset Q)$	Ł$_3$D1, P/P, Q/Q	
2.	$\sim P \supset \sim\sim(P \supset Q)$	1 D.N.	
3.	$\sim(P \supset Q) \supset P$	2 Con.	(p. 276)

GHS (Generalised Hypothetical Syllogism) From $(P_1 \supset (P_2 \supset \cdots \supset (P_{n-1} \supset P_n) \ldots)$ and $P_n \supset Q$, infer $(P_1 \supset (P_2 \supset \cdots \supset P_{n-1} \supset Q) \ldots)$

<u>Base case</u>: n = 3

We derive $P_1 \supset (P_2 \supset Q)$ from $P_1 \supset (P_2 \supset P_3)$ and $P_3 \supset Q$

1.	$P_1 \supset (P_2 \supset P_3)$	given	
2.	$P_3 \supset Q$	given	
3.	$(P_2 \supset P_3) \supset ((P_3 \supset Q) \supset (P_2 \supset Q))$	Ł$_3$2, P_2/P, P_3/Q, Q/R	
4.	$(P_3 \supset Q) \supset ((P_2 \supset P_3) \supset (P_2 \supset Q))$	3 Tran.	
5.	$(P_2 \supset P_3) \supset (P_2 \supset Q)$	2,4 M.P.	
6.	$P_1 \supset (P_2 \supset Q)$	1,5 H.S.	(p. 276)

<u>Hypothesis</u>:

From $(P'_1 \supset (P'_2 \supset \cdots \supset (P'_{n-2} \supset P'_{n-1}) \ldots)$ and $P'_{n-1} \supset Q'$

infer $(P'_1 \supset (P'_2 \supset \cdots \supset (P'_{n-2} \supset Q') \ldots)$

<u>Induction step</u>:

Step n - 1 to n: for arbitrary n > 3, the derivation begins as

1.	$P_1 \supset (P_2 \supset (P_3 \supset \cdots \supset (P_{n-1} \supset P_n) \ldots))$	given

2. $P_n \supset Q$ given

3. $(P_{n-1} \supset P_n) \supset ((P_n \supset Q) \supset (P_{n-1} \supset Q))$ Ł$_3$2, P_{n-1}/P, P_n/Q, Q/R

4. $((P_n \supset Q) \supset ((P_{n-1} \supset P_n) \supset (P_{n-1} \supset Q))$ 3 Tran.

5. $(P_{n-1} \supset P_n) \supset (P_{n-1} \supset Q)$ 4,2 M.P. (p. 278)

The formulae on lines 1 and 5 are, respectively, instances of the premises:

$(P'_1 \supset ((P'_2 \supset (P'_3 \supset \cdots \supset (P'_{n-2} \supset P'_{n-1}) \ldots)$ and $P'_{n-1} \supset Q'$ of our induction hypothesis with

- $P_{n-1} \supset P_n$ in place of P'_{n-1} and
- $P_{n-1} \supset Q$ in place of Q' and
- P_i in place of P'_i for $1 \leq i \leq n$ - 2

Fronhöfer completes the derivation with the instantiated conclusion of the induction hypothesis:

6. $P_1 \supset ((P_2 \supset (P_3 \supset \cdots \supset (P_{n-1} \supset Q) \ldots)$ 1,5 I.H., $P_{n-1} \supset P_n/P'_{n-1}$, $P_{n-1} \supset Q/\, Q'$, $P_i/\, P'_i$ for $1 \leq i \leq n$ - 2 (p. 278)

GMP (Generalised Modus Ponens) From $(P_1 \supset (P_2 \supset \cdots \supset (P_{n-1} \supset P_n) \ldots)$ and one of the antecedents P_i, $1 \leq i \leq n-1$, infer the conditional that results from deleting P_i, the conditional hook following P_i, and associated parentheses.

Justification

- By repeated application of Tran. the antecedents $\mathrm{P}_1, \ldots .\, \mathrm{P}_{n-1}$ can be permuted in any order.
- Specifically, P_i can be moved to the beginning of the formula, leaving the order of the other antecedents unchanged.
- Then a single application of M.P. will produce the desired formula with P_i removed. (p. 278)

Constructive Dilemma

In classical logic the following argument is valid:

$P \supset Q$
$\sim P \supset Q$
$\therefore Q$

(cyan rows below) and the corresponding rule is derivable in CLA. However the inference is not valid in Ł$_3$ if both P and Q have the value $\odot$ (magenta rows):

((P	⊃	Q)	(~	P	⊃	Q))	Q
T	T	T	F	T	T	T	T
T	$\odot$	$\odot$	F	T	T	$\odot$	$\odot$
T	F	F	F	T	T	F	F
$\odot$	T	T	$\odot$	$\odot$	T	T	T
$\odot$	T	$\odot$	$\odot$	$\odot$	T	$\odot$	$\odot$
$\odot$	$\odot$	F	$\odot$	$\odot$	$\odot$	F	F
F	T	T	T	F	T	T	T
F	T	$\odot$	T	F	$\odot$	$\odot$	$\odot$
F	T	F	T	F	F	F	F

Modified Constructive Dilemma

The argument below, however is valid in Ł$_3$:

$P \supset Q$
$(P \supset \sim P) \supset Q$
$\therefore Q$

and the corresponding rule is derivable in Ł$_3$A, green rows below.

((P	⊃	Q)	((P	⊃	~	P)	⊃	Q))	Q
T	T	T	T	F	F	T	T	T	T
T	$\odot$	$\odot$	T	F	F	T	T	$\odot$	$\odot$
T	F	F	T	F	F	T	T	F	F
$\odot$	T	T	$\odot$	T	$\odot$	$\odot$	T	T	T
$\odot$	T	$\odot$	$\odot$	T	$\odot$	$\odot$	$\odot$	$\odot$	$\odot$
$\odot$	$\odot$	F	$\odot$	T	$\odot$	$\odot$	F	F	F
F	T	T	F	T	T	F	T	T	T
F	T	$\odot$	F	T	T	F	$\odot$	$\odot$	$\odot$
F	T	F	F	T	T	F	F	F	F

MCD (Modified Constructive Dilemma) From $P \supset Q$ and $(P \supset \sim P) \supset Q$, infer Q.

1. $P \supset Q$ — given
2. $(P \supset \sim P) \supset Q$ — given
3. $(P \supset Q) \supset ((Q \supset \sim P) \supset (P \supset \sim P))$ — Ł$_3$2, P/P, Q/Q, $\sim P/R$
4. $(Q \supset \sim P) \supset (P \supset \sim P)$ — 3,1 M.P.

5. $(Q \supset \sim P) \supset Q$ 4,2 H.S.

6. $(Q \supset \sim Q) \supset ((\sim Q \supset \sim P) \supset (Q \supset \sim P))$ Ł$_3$2, Q/P, $\sim Q/Q$, $\sim P/R$

7. $(\sim Q \supset \sim P) \supset ((Q \supset \sim Q) \supset (Q \supset \sim P))$ 6 Tran.

8. $\sim Q \supset \sim P$ 1 GCon.

9. $(Q \supset \sim Q) \supset (Q \supset \sim P)$ 7,8 M.P.

10. $(Q \supset \sim Q) \supset Q$ 9,5 H.S.

11. $((Q \supset \sim Q) \supset Q) \supset Q$ Ł$_3$4, Q/P

12. Q 11,10 M.P.

Rem. 4.43 MCD can also be expressed as: from $P \supset Q$ and $\sim_{\mathrm{BE}} P \supset Q$, infer Q. (p. 282)

DE (Disjunction Elimination) From $P \vee Q$, $P \supset R$ and $Q \supset R$, infer R.

1. $(P \supset Q) \supset Q$ given (rewritten from $P \vee Q$)

2. $P \supset R$ given

3. $Q \supset R$ given

4. $\sim(P \supset Q) \supset P$ Ł$_3$D8, P/P, Q/Q

5. $\sim(P \supset Q) \supset R$ 4,2 H.S.

6. $(\sim\sim(P \supset Q) \supset \sim\sim Q) \supset (\sim Q \supset \sim(P \supset Q))$ Ł$_3$3, $\sim(P \supset Q)/P$, $\sim Q/Q$

7. $((P \supset Q) \supset Q) \supset (\sim Q \supset \sim(P \supset Q))$ 6 D.N. (twice)

8. $((P \supset Q) \supset Q) \supset (\sim Q \supset P)$ 7,4 GHS

9. $(\sim Q \supset P) \supset ((P \supset \sim P) \supset (\sim Q \supset \sim P))$ Ł$_3$2, $\sim Q/P$, P/Q, $\sim P/R$

10. $(\sim Q \supset \sim P) \supset (P \supset Q)$ Ł$_3$3, Q/P, P/Q

11. $(\sim Q \supset P) \supset ((P \supset \sim P) \supset (P \supset Q))$ 9,10 GHS

12. $(P \supset Q) \supset$ $(((P \supset Q) \supset \sim(P \supset Q)) \supset \sim(P \supset Q))$ — Ł$_3$D6, $P \supset Q/P$ $\sim(P \supset Q)/Q$

13. $(\sim Q \supset P) \supset ((P \supset \sim P) \supset$ $(((P \supset Q) \supset \sim(P \supset Q)) \supset \sim(P \supset Q)))$ — 11,12 GHS

14. $(\sim Q \supset P) \supset ((P \supset \sim P) \supset$ $(((P \supset Q) \supset \sim(P \supset Q)) \supset R))$ — 13,5 GHS

15. $((P \supset Q) \supset Q) \supset ((P \supset \sim P) \supset$ $(((P \supset Q) \supset \sim(P \supset Q)) \supset R))$ — 14,8 H.S.

16. $((P \supset Q) \supset Q) \supset (((P \supset Q) \supset$ $\sim(P \supset Q)) \supset ((P \supset \sim P) \supset R))$ — 15 Tran.

17. $((P \supset Q) \supset \sim(P \supset Q)) \supset$ $(((P \supset Q) \supset Q) \supset ((P \supset \sim P) \supset R))$ — 16 Tran.

18. $((P \supset Q) \supset \sim(P \supset Q)) \supset$ $((P \supset \sim P) \supset (((P \supset Q) \supset Q) \supset R))$ — 17. Tran.

19. $(P \supset Q) \supset (((P \supset Q) \supset Q) \supset Q)$ — Ł$_3$D6, $P \supset Q/P$, Q/Q

20. $(P \supset Q) \supset (((P \supset Q) \supset Q) \supset R)$ — 19,3 GHS

21. $(((P \supset Q) \supset Q) \supset R) \supset$ $((P \supset \sim P) \supset (((P \supset Q) \supset Q) \supset R))$ — Ł$_3$1, $((P \supset Q) \supset Q) \supset R/P$ $P \supset \sim P/Q$

22. $(P \supset Q) \supset ((P \supset \sim P) \supset$ $(((P \supset Q) \supset Q) \supset R))$ — 21,20 H.S.

23. $(P \supset \sim P) \supset (((P \supset Q) \supset Q) \supset R)$ — 18,22 MCD

24. $R \supset (((P \supset Q) \supset Q) \supset R)$ — Ł$_3$1, R/P, $(P \supset Q) \supset Q/Q$

25. $P \supset (((P \supset Q) \supset Q) \supset R)$ — 24,2 H.S.

26. $((P \supset Q) \supset Q) \supset R$ — 23,25 MCD

27. R — 26,1 M.P. (p. 282-284)

DC (Disjunctive Consequence) From $P \supset R$ and $Q \supset R$, infer $(P \vee Q) \supset R$.

According to Fronhöfer this is implicit in lines 2 - 26 of the previous derivation.

Ł$_3$**D9**: $(P \text{ v } Q) \supset (Q \text{ v } P)$ can be rewritten as $((P \supset Q) \supset Q) \supset ((Q \supset P) \supset P))$

1. $P \supset ((Q \supset P) \supset P)$ — Ł$_3$1, P/P, $Q \supset P/Q$
2. $Q \supset ((Q \supset P) \supset P)$ — Ł$_3$D6, Q/P, P/Q
3. $((P \supset Q) \supset Q) \supset ((Q \supset P) \supset P)$ — 1,2 DC (p. 284)

Ł$_3$**D10** $(P \supset Q) \text{ v } (Q \supset P)$ can be rewritten as $((P \supset Q) \supset (Q \supset P)) \supset (Q \supset P)$

1. $((P \supset Q) \supset (Q \supset P)) \supset$
   $(((Q \supset P) \supset P) \supset ((P \supset Q) \supset P))$ — Ł$_3$2, $P \supset Q/P$
   $Q \supset P/Q$, P/R
2. $((P \supset Q) \supset Q) \supset ((Q \supset P) \supset P)$ — Ł$_3$D9, P/P, Q/Q
3. $((Q \supset P) \supset P) \supset ((P \supset Q) \supset Q)$ — Ł$_3$D9, Q/P, P/Q
4. $((P \supset Q) \supset (Q \supset P)) \supset$
   $(((P \supset Q) \supset Q) \supset ((P \supset Q) \supset P))$ — 1,2,3 Sub.
5. $(((P \supset Q) \supset (Q \supset P)) \supset$
   $(((P \supset Q) \supset Q) \supset ((P \supset Q) \supset P))) \supset$
   $(((((P \supset Q) \supset Q) \supset ((P \supset Q) \supset P)) \supset$
   $((Q \supset (P \supset Q)) \supset (Q \supset P))) \supset$
   $(((P \supset Q) \supset (Q \supset P)) \supset$
   $((Q \supset (P \supset Q)) \supset (Q \supset P))))$ — Ł$_3$2, $(P \supset Q) \supset (Q \supset P)/P$,
   $((P \supset Q) \supset Q) \supset$
   $((P \supset Q) \supset P)/Q$,
   $(Q \supset (P \supset Q)) \supset$
   $(Q \supset P)/R$
6. $(((P \supset Q) \supset Q) \supset ((P \supset Q) \supset P)) \supset$
   $((Q \supset (P \supset Q)) \supset (Q \supset P)) \supset$
   $(((P \supset Q) \supset (Q \supset P)) \supset$
   $((Q \supset (P \supset Q)) \supset (Q \supset P)))$ — 5,4 M.P.
7. $((\sim Q \supset \sim(P \supset Q)) \supset (\sim P \supset \sim(P \supset Q))) \supset$
   $((\sim Q \supset \sim(P \supset Q)) \supset (\sim P \supset \sim(P \supset Q)))$ — Ł$_3$D4, $(\sim Q \supset \sim(P \supset Q)) \supset$
   $(\sim P \supset \sim(P \supset Q))/P$
8. $((\sim Q \supset \sim(P \supset Q)) \supset (\sim P \supset \sim(P \supset Q))) \supset$
   $(\sim P \supset ((\sim Q \supset \sim(P \supset Q)) \supset \sim(P \supset Q)))$ — 7 Tran.
9. $((\sim Q \supset \sim(P \supset Q)) \supset \sim(P \supset Q)) \supset$
   $((\sim(P \supset Q) \supset \sim Q) \supset \sim Q)$ — Ł$_3$D9, $\sim Q/P$, $\sim(P \supset Q)/Q$
10. $((\sim Q \supset \sim(P \supset Q)) \supset (\sim P \supset \sim(P \supset Q))) \supset$
    $(\sim P \supset ((\sim(P \supset Q) \supset \sim Q) \supset \sim Q))$ — 8,9 GHS
11. $((\sim Q \supset \sim(P \supset Q)) \supset (\sim P \supset \sim(P \supset Q))) \supset$ — 10 Tran.

$((\sim(P \supset Q) \supset \sim Q) \supset (\sim P \supset \sim Q))$

12. $((P \supset Q) \supset Q) \supset ((P \supset Q) \supset P) \supset ((Q \supset (P \supset Q)) \supset (Q \supset P))$	11 GCon. (four times)	
13. $((P \supset Q) \supset (Q \supset P)) \supset ((Q \supset (P \supset Q)) \supset (Q \supset P))$	12,4 H.S.	
14. $((Q \supset (P \supset Q)) \supset (((P \supset Q) \supset (Q \supset P)) \supset (Q \supset P)))$	13 Tran.	
15. $Q \supset (P \supset Q)$	Ł$_3$1, Q/P, P/Q	
16. $((P \supset Q) \supset (Q \supset P)) \supset (Q \supset P)$	14,15 M.P.	(p. 286)

Ł$_3$D11: $(P \supset (P \supset (P \supset Q))) \supset (P \supset (P \supset Q))$

1. $\sim P \supset (P \supset Q)$	Ł$_3$D1, P/P, Q/Q	
2. $(P \supset \sim P) \supset ((\sim P \supset (P \supset Q)) \supset (P \supset (P \supset Q)))$	Ł$_3$2, P/P, $\sim P/Q$, $P \supset Q/R$	
3. $(P \supset \sim P) \supset (P \supset (P \supset Q))$	1,2 GMP	
4. $((P \supset \sim P) \supset (P \supset (P \supset Q))) \supset (((P \supset (P \supset Q)) \supset P) \supset ((P \supset \sim P) \supset P))$	Ł$_3$2, $P \supset \sim P/P$, $P \supset (P \supset Q)/Q$, P/R	
5. $((P \supset (P \supset Q)) \supset P) \supset ((P \supset \sim P) \supset P)$	4,3 M.P.	
6. $((P \supset \sim P) \supset P) \supset P$	Ł$_3$4, P/P	
7. $((P \supset (P \supset Q)) \supset P) \supset P$	5,6 H.S.	
8. $(((P \supset (P \supset Q)) \supset P) \supset P) \supset ((P \supset (P \supset (P \supset Q))) \supset (P \supset (P \supset Q)))$	Ł$_3$D9, $P \supset (P \supset Q)/P$, P/Q	
9. $(P \supset (P \supset (P \supset Q))) \supset (P \supset (P \supset Q))$	8,7 M.P.	(p. 288)

C.I. (Conjunction Introduction) From P and Q, infer $P \bullet Q$.

1. P	CPA
2. Q	CPA

3.	$\sim\sim P$	D.N.
4.	$\sim\sim P \supset (\sim P \supset \sim Q)$	Ł$_3$D1, $\sim P/P, \sim Q/Q$
5.	$\sim P \supset \sim Q$	3,4 M.P.
6.	$(\sim P \supset \sim Q) \supset (((\sim P \supset \sim Q) \supset \sim Q) \supset \sim Q$	Ł$_3$D6, $\sim P \supset \sim Q/P, \sim Q/Q$
7.	$((\sim P \supset \sim Q) \supset \sim Q) \supset \sim Q$	5,6 M.P.
8.	$\sim\sim Q \supset \sim((\sim P \supset \sim Q) \supset \sim Q)$	7 GCon.
9.	$\sim\sim Q$	2 D.N.
10.	$\sim((\sim P \supset \sim Q) \supset \sim Q)$	8,9 M.P. (rewritten from $P \bullet Q$) (p. 288)

Rem. 4.44 Note $(P \supset (P \supset Q)) \supset (P \supset Q)$ does not hold in Ł$_3$, see magenta row below. However it does correspond to (left) contraction in subsequent systems.

(P	⊃	(P	⊃	Q))	⊃	(P	⊃	Q)
T	T	T	T	T	T	T	T	T
T	⊙	T	⊙	⊙	T	T	⊙	⊙
T	F	T	F	F	T	T	F	F
⊙	T	⊙	T	T	T	⊙	T	T
⊙	T	⊙	T	⊙	T	⊙	T	⊙
⊙	T	⊙	⊙	F	⊙	⊙	⊙	F
F	T	F	T	T	T	F	T	T
F	T	F	T	⊙	T	F	T	⊙
F	T	F	T	F	T	F	T	F

Rem. 4.45 Note also, $(\sim P \supset P) \supset P$ does not hold in Ł$_3$, see magenta row below. However it does correspond to (right) contraction in subsequent systems.

(~	P	⊃	P)	⊃	P
F	T	T	T	T	T
⊙	⊙	T	⊙	⊙	⊙
T	F	F	F	T	F

Rem. 4.46 The deduction theorem

if $\mathcal{S} \cup \{F\} \vdash G$, then $\mathcal{S} \vdash F \supset G$

does not hold in Ł$_3$A.

***E.g.* 4.47** Whenever $P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R)))$ is true in Ł$_3$ then so is R, however $P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R))) \supset R$ is not a Ł$_3$ tautology.

Proof: R may be derived from $P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R)))$ as follows

1. $P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R)))$ CPA
2. P 1 LSimp.
3. $((P \supset Q) \bullet (P \supset (Q \supset R)))$ 1 RSimp.
4. $P \supset Q$ 3 LSimp.
5. $P \supset (Q \supset R)$ 3 RSimp.
6. $Q \supset R$ 5,2 M.P.
7. $P \supset R$ 4,6 H.S.
8. R 7,2 M.P.

(P	•	((P	⊃	Q)	•	(P	⊃	(Q	⊃	R))))	⊃	R
T	T	T	T	T	T	T	T	T	T	T	T	T
T	⊙	T	T	T	⊙	T	⊙	T	⊙	⊙	T	⊙
T	F	T	T	T	F	T	F	T	F	F	T	F
T	⊙	T	⊙	⊙	⊙	T	T	⊙	T	T	T	T
T	⊙	T	⊙	⊙	⊙	T	T	⊙	T	⊙	T	⊙
T	⊙	T	⊙	⊙	⊙	T	⊙	⊙	⊙	F	⊙	F
T	F	T	F	F	F	T	T	F	T	T	T	T
T	F	T	F	F	F	T	T	F	T	⊙	T	⊙
T	F	T	F	F	F	T	T	F	T	F	T	F
⊙	⊙	⊙	T	T	T	⊙	T	T	T	T	T	T
⊙	⊙	⊙	T	T	T	⊙	T	T	⊙	⊙	T	⊙
⊙	⊙	⊙	T	T	⊙	⊙	⊙	T	F	F	⊙	F
⊙	⊙	⊙	T	⊙	T	⊙	T	⊙	T	T	T	T
⊙	⊙	⊙	T	⊙	T	⊙	T	⊙	T	⊙	T	⊙
⊙	⊙	⊙	T	⊙	T	⊙	T	⊙	⊙	F	⊙	F
⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	T	T	T
⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	⊙	T	⊙
⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	F	⊙	F
F	F	F	T	T	T	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T	T	⊙	⊙	T	⊙
F	F	F	T	T	T	F	T	T	F	F	T	F
F	F	F	T	⊙	T	F	T	⊙	T	T	T	T
F	F	F	T	⊙	T	F	T	⊙	T	⊙	T	⊙
F	F	F	T	⊙	T	F	T	⊙	⊙	F	T	F
F	F	F	T	F	T	F	T	F	T	T	T	T
F	F	F	T	F	T	F	T	F	T	⊙	T	⊙
F	F	F	T	F	T	F	T	F	T	F	T	F

Colour Key: R is true when $P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R)))$ is true, green but where $P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R))) \supset R$ is not true in Ł$_3$, magenta.

(p. 292)

Stutterer's Deduction Theorem

Lem. 4.48 According to Fronhöfer, the following modified deduction theorem obtains:

For a set of formulae $\mathcal{S}$ of Ł$_3$, and formulae $\mathbf{A}$ and $\mathbf{B}$ also of Ł$_3$

$\mathcal{S} \cup \{\mathbf{A}\} \vdash \mathbf{B} \equiv \mathcal{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset \mathbf{B})$

Proof: We apply M.P. twice.

Suppose that the sequence of formulae $\mathbf{C}_1, \mathbf{C}_2, \dots \mathbf{C}_p$ constitutes a derivation of $\mathbf{B}$ from $\mathcal{S} \cup \{\mathbf{A}\}$.

We establish by mathematical induction on i that $\mathcal{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i)$ obtains for each i from 1 to p, and hence that $\mathcal{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset \mathbf{B})$ obtains.

- In the case that $\mathbf{C}_i$ is $\mathbf{A}$, then $\mathcal{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i)$ by Lemma 4.9(c), since

  $\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i)$ is $\mathbf{A} \supset (\mathbf{A} \supset \mathbf{A})$ is Ł$_3$1 with $\mathbf{A}/\mathbf{P}$ and $\mathbf{A}/\mathbf{Q}$.

- In the case that $\mathbf{C}_i$ is an axiom or a member of $\mathcal{S}$, then

  (a) $\mathcal{S} \vdash \mathbf{C}_i$ by Lemma 4.9(c) or by Lemma 4.9(e) as the case may be, but

  (b) $\mathcal{S} \vdash \mathbf{C}_i \supset (\mathbf{A} \supset \mathbf{C}_i)$ by Lemma 4.9(c), since $\mathbf{C}_i \supset (\mathbf{A} \supset \mathbf{C}_i)$ is Ł$_3$1 with $\mathbf{C}_i/\mathbf{P}$ and $\mathbf{A}/\mathbf{Q}$. Hence,

  (c) $\mathcal{S} \vdash \mathbf{A} \supset \mathbf{C}_i$ by Lemma 4.9(f) (M.P. (a),(b)). But,

  (d) $\mathcal{S} \vdash (\mathbf{A} \supset \mathbf{C}_i) \supset (\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i))$ by Lemma 4.9(c), since $(\mathbf{A} \supset \mathbf{C}_i) \supset (\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i))$ is Ł$_3$1 with $\mathbf{A} \supset \mathbf{C}_i/\mathbf{P}$ and $\mathbf{A}/\mathbf{Q}$. Hence,

  (e) $\mathcal{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i)$ by Lemma 4.9(f) (M.P. (c),(d)).

- In the case that $\mathbf{C}_i$ is obtained by M.P. from $\mathbf{C}_h \supset \mathbf{C}_i$ and $\mathbf{C}_h$, then

  $\mathcal{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_h)$ and $\mathcal{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset (\mathbf{C}_h \supset \mathbf{C}_i)$ I.H.

  n $\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_h)$
  ⋮
  m $\mathbf{A} \supset (\mathbf{A} \supset (\mathbf{C}_h \supset \mathbf{C}_i))$

  m+1 $\mathbf{A} \supset (\mathbf{C}_h \supset (\mathbf{A} \supset \mathbf{C}_i))$ m Tran.

m+2 $\mathbf{C}_h \supset (\boldsymbol{A} \supset (\boldsymbol{A} \supset \mathbf{C}_i))$ — m+1 Tran.

m+3 $\mathbf{A} \supset (\mathbf{A} \supset (\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i)))$ — m+2, n GHS

m+4 $(\mathbf{A} \supset (\mathbf{A} \supset (\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i)))) \supset (\mathbf{A} \supset (\mathbf{A} \supset (\mathbf{A} \supset \boldsymbol{C}_i)))$ — Ł$_3$D11, **A/P**, $\mathbf{A} \supset \mathbf{C}_i/\mathbf{Q}$

m+5 $\mathbf{A} \supset (\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i))$ — m+4, m+3 M.P.

m+6 $(\mathbf{A} \supset (\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i))) \supset (\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i))$ — Ł$_3$D11**, A/P,** $\mathbf{C}_i/\mathbf{Q}$

m+7 $\mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i)$ — m+6, m+5 M.P.

Hence $\mathcal{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset \mathbf{C}_i)$ by Lemma 4.9. p. 296

***E.g.* 4.49** From *e.g.* 4.47, R is derivable from $P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R)))$ in Ł$_3$A. Therefore it follows from the Modified Deduction Theorem that $(P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R)))) \supset ((P \bullet ((P \supset Q) \bullet (P \supset (Q \supset R)))) \supset R)$ is a theorem. Verify by truth table below.

(P	•	((P	⊃	Q)	•	(P	⊃	(Q	⊃	R))))	⊃	((P	•	((P	⊃	Q)	•	(P	⊃	(Q	⊃	R))))	⊃	R)
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	⊙	T	T	T	⊙	T	⊙	T	⊙	⊙	T	T	⊙	T	T	T	⊙	T	⊙	T	⊙	⊙	T	⊙
T	F	T	T	T	F	T	F	T	F	F	T	T	F	T	T	T	F	T	F	T	F	F	T	F
T	⊙	T	⊙	⊙	⊙	T	T	⊙	T	T	T	T	⊙	T	⊙	⊙	⊙	T	T	⊙	T	T	T	T
T	⊙	T	⊙	⊙	⊙	T	T	⊙	T	⊙	T	T	⊙	T	⊙	⊙	⊙	T	T	⊙	T	⊙	T	⊙
T	⊙	T	⊙	⊙	⊙	T	⊙	⊙	⊙	F	T	T	⊙	T	⊙	⊙	⊙	T	⊙	⊙	⊙	F	⊙	F
T	F	T	F	F	F	T	T	F	T	T	T	T	F	T	F	F	F	T	T	F	T	T	T	T
T	F	T	F	F	F	T	T	F	T	⊙	T	T	F	T	F	F	F	T	T	F	T	⊙	T	⊙
T	F	T	F	F	F	T	T	F	T	F	T	T	F	T	F	F	F	T	T	F	T	F	T	F
⊙	⊙	⊙	T	T	T	⊙	T	T	T	T	T	⊙	⊙	⊙	T	T	T	⊙	T	T	T	T	T	T
⊙	⊙	⊙	T	T	T	⊙	T	T	⊙	⊙	T	⊙	⊙	⊙	T	T	T	⊙	T	T	⊙	⊙	T	⊙
⊙	⊙	⊙	T	T	⊙	⊙	⊙	T	F	F	T	⊙	⊙	⊙	T	T	⊙	⊙	⊙	T	F	F	⊙	F
⊙	⊙	⊙	T	⊙	T	⊙	T	⊙	T	T	T	⊙	⊙	⊙	T	⊙	T	⊙	T	⊙	T	T	T	T
⊙	⊙	⊙	T	⊙	T	⊙	T	⊙	T	⊙	T	⊙	⊙	⊙	T	⊙	T	⊙	T	⊙	T	⊙	T	⊙
⊙	⊙	⊙	T	⊙	⊙	⊙	T	⊙	⊙	F	T	⊙	⊙	⊙	T	⊙	T	⊙	T	⊙	⊙	F	⊙	F
⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	T	T	⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	T	T	T
⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	⊙	T	⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	⊙	T	⊙
⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	F	T	⊙	⊙	⊙	⊙	F	⊙	⊙	T	F	T	F	⊙	F
F	F	F	T	T	T	F	T	T	T	T	T	F	F	F	T	T	T	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T	T	⊙	⊙	T	F	F	F	T	T	T	F	T	T	⊙	⊙	T	⊙
F	F	F	T	T	T	F	T	T	F	F	T	F	F	F	T	T	T	F	T	T	F	F	T	F
F	F	F	T	⊙	T	F	T	⊙	T	T	T	F	F	F	T	⊙	T	F	T	⊙	T	T	T	T
F	F	F	T	⊙	T	F	T	⊙	T	⊙	T	F	F	F	T	⊙	T	F	T	⊙	T	⊙	T	⊙
F	F	F	T	⊙	T	F	T	⊙	⊙	F	T	F	F	F	T	⊙	T	F	T	⊙	⊙	F	T	F
F	F	F	T	F	T	F	T	F	T	T	T	F	F	F	T	F	T	F	T	F	T	T	T	T
F	T	F	T	F	T	F	T	F	T	⊙	T	F	F	F	T	F	T	F	T	F	T	⊙	T	⊙
F	F	F	T	F	T	F	T	F	T	F	T	F	F	F	T	F	T	F	T	F	T	F	T	F

(p. 298)

Ł_3**D11** $(P \,\&\, (P \supset Q)) \supset Q$

1. $(P \supset Q) \supset (P \supset Q)$ — Ł_3D4, $(P \supset Q)/P$
2. $(P \supset Q) \supset ({\sim}{\sim}P \supset {\sim}{\sim}Q)$ — 1 D.N. (twice)
3. $(P \supset Q) \supset ({\sim}Q \supset {\sim}P)$ — 2 GCon.
4. ${\sim}Q \supset ((P \supset Q) \supset {\sim}P)$ — 3 Tran.
5. ${\sim}Q \supset ({\sim}{\sim}(P \supset Q) \supset {\sim}P)$ — 4 D.N.
6. ${\sim}Q \supset (P \supset {\sim}(P \supset Q))$ — 5 GCon.
7. ${\sim}Q \supset {\sim}{\sim}(P \supset {\sim}(P \supset Q))$ — 6 D.N.
8. ${\sim}(P \supset {\sim}(P \supset Q)) \supset Q$ — 7 GCon.

The last line is equivalent to $(P \,\&\, (P \supset Q)) \supset Q$ in terms of $\supset$ and $\sim$.
Recall that $(P \bullet (P \supset Q)) \supset Q$ is not a tautology in Ł_3, *cf*. Rem 2.28. (p. 298)

Ł_3**D16** $(P \supset Q) \supset ((R \supset S) \supset ((P \,\&\, R) \supset (Q \,\&\, S)))$

Rothenberg, (2005, A.21) does provide a derivation of the above derived rule of inference; however his justification of the first two lines of his derivation refer to an earlier derivation of his, which in turn relies on an earlier definition in his thesis, neither of which form part of Fronhöfer's presentation. Instead, we rely on a "brute force" proof in the form of a truth table, by which we can prove this a tautology. We have had to split the truth table below into three – the first for P always T, the second for P always $\odot$, and the third for P always F.

(P	⊃	Q)	⊃	((R	⊃	S)	⊃	((P	&	R)	⊃	(Q	&	S)))
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	T	T	T	⊙	⊙	T	T	T	T	⊙	T	⊙	⊙
T	T	T	T	T	F	F	T	T	T	T	F	T	F	F
T	T	T	T	⊙	T	T	T	T	⊙	⊙	T	T	T	T
T	T	T	T	⊙	T	⊙	T	T	⊙	⊙	T	T	⊙	⊙
T	T	T	T	⊙	⊙	F	T	T	⊙	⊙	⊙	T	F	F
T	T	T	T	F	T	T	T	T	F	F	T	T	T	T
T	T	T	T	F	T	⊙	T	T	F	F	T	T	⊙	⊙
T	T	T	T	F	T	F	T	T	F	F	T	T	F	F
T	⊙	⊙	T	T	T	T	⊙	T	T	T	⊙	⊙	⊙	T
T	⊙	⊙	T	T	⊙	⊙	⊙	T	T	T	F	⊙	F	⊙
T	⊙	⊙	T	T	F	F	T	T	T	T	F	⊙	F	F
T	⊙	⊙	T	⊙	T	T	T	T	⊙	⊙	T	⊙	⊙	T
T	⊙	⊙	T	⊙	T	⊙	⊙	T	⊙	⊙	⊙	⊙	F	⊙
T	⊙	⊙	T	⊙	⊙	F	T	T	⊙	⊙	⊙	⊙	F	F
T	⊙	⊙	T	F	T	T	T	T	F	F	T	⊙	⊙	T
T	⊙	⊙	T	F	T	⊙	T	T	F	F	T	⊙	F	⊙
T	⊙	⊙	T	F	T	F	T	T	F	F	T	⊙	F	F
T	F	F	T	T	T	T	F	T	T	T	F	F	F	T
T	F	F	T	T	⊙	⊙	⊙	T	T	T	F	F	F	⊙
T	F	F	T	T	F	F	T	T	T	T	F	F	F	F
T	F	F	T	⊙	T	T	⊙	T	⊙	⊙	⊙	F	F	T
T	F	F	T	⊙	T	⊙	⊙	T	⊙	⊙	⊙	F	F	⊙
T	F	F	T	⊙	⊙	F	T	T	⊙	⊙	⊙	F	F	F
T	F	F	T	F	T	T	T	T	F	F	T	F	F	T
T	F	F	T	F	T	⊙	T	T	F	F	T	F	F	⊙
T	F	F	T	F	T	F	T	T	F	F	T	F	F	F

(P	⊃	Q)	⊃	((R	⊃	S)	⊃	((P	&	R)	⊃	(Q	&	S)))
⊙	T	T	T	T	T	T	T	⊙	⊙	T	T	T	T	T
⊙	T	T	T	T	⊙	⊙	T	⊙	⊙	T	T	T	⊙	⊙
⊙	T	T	T	T	F	F	T	⊙	⊙	T	⊙	T	F	F
⊙	T	T	T	⊙	T	T	T	⊙	F	⊙	T	T	T	T
⊙	T	T	T	⊙	T	⊙	T	⊙	F	⊙	T	T	⊙	⊙
⊙	T	T	T	⊙	⊙	F	T	⊙	F	⊙	T	T	F	F
⊙	T	T	T	F	T	T	T	⊙	F	F	T	T	T	T
⊙	T	T	T	F	T	⊙	T	⊙	F	F	T	T	⊙	⊙
⊙	T	T	T	F	T	F	T	⊙	F	F	T	T	F	F
⊙	T	⊙	T	T	T	T	T	⊙	⊙	T	T	⊙	⊙	T
⊙	T	⊙	T	T	⊙	⊙	T	⊙	⊙	T	⊙	⊙	F	⊙
⊙	T	⊙	T	T	F	F	T	⊙	⊙	T	⊙	⊙	F	F
⊙	T	⊙	T	⊙	T	T	T	⊙	F	⊙	T	⊙	⊙	T
⊙	T	⊙	T	⊙	T	⊙	T	⊙	F	⊙	T	⊙	F	⊙
⊙	T	⊙	T	⊙	⊙	F	T	⊙	F	⊙	T	⊙	F	F
⊙	T	⊙	T	F	T	T	T	⊙	F	F	T	⊙	⊙	T
⊙	T	⊙	T	F	T	⊙	T	⊙	F	F	T	⊙	F	⊙
⊙	T	⊙	T	F	T	F	T	⊙	F	F	T	⊙	F	F
⊙	⊙	F	T	T	T	T	⊙	⊙	⊙	T	⊙	F	F	T
⊙	⊙	F	T	T	⊙	⊙	T	⊙	⊙	T	⊙	F	F	⊙
⊙	⊙	F	T	T	F	F	T	⊙	⊙	T	⊙	F	F	F
⊙	⊙	F	T	⊙	T	T	T	⊙	F	⊙	T	F	F	T
⊙	⊙	F	T	⊙	T	⊙	T	⊙	F	⊙	T	F	F	⊙
⊙	⊙	F	T	⊙	⊙	F	T	⊙	F	⊙	T	F	F	F
⊙	⊙	F	T	F	T	T	T	⊙	F	F	T	F	F	T
⊙	⊙	F	T	F	T	⊙	T	⊙	F	F	T	F	F	⊙
⊙	⊙	F	T	F	T	F	T	⊙	F	F	T	F	F	F

(P	⊃	Q)	⊃	((R	⊃	S)	⊃	((P	&	R)	⊃	(Q	&	S)))
F	T	T	T	T	T	T	T	F	F	T	T	T	T	T
F	T	T	T	T	⊙	⊙	T	F	F	T	T	T	⊙	⊙
F	T	T	T	T	F	F	T	F	F	T	T	T	F	F
F	T	T	T	⊙	T	T	T	F	F	⊙	T	T	T	T
F	T	T	T	⊙	T	⊙	T	F	F	⊙	T	T	⊙	⊙
F	T	T	T	⊙	⊙	F	T	F	F	⊙	T	T	F	F
F	T	T	T	F	T	T	T	F	F	F	T	T	T	T
F	T	T	T	F	T	⊙	T	F	F	F	T	T	⊙	⊙
F	T	T	T	F	T	F	T	F	F	F	T	T	F	F
F	T	⊙	T	T	T	T	T	F	F	T	T	⊙	⊙	T
F	T	⊙	T	T	⊙	⊙	T	F	F	T	T	⊙	F	⊙
F	T	⊙	T	T	F	F	T	F	F	T	T	⊙	F	F
F	T	⊙	T	⊙	T	T	T	F	F	⊙	T	⊙	F	T
F	T	⊙	T	⊙	T	⊙	T	F	F	⊙	T	⊙	F	⊙
F	T	⊙	T	⊙	⊙	F	T	F	F	⊙	T	⊙	F	F
F	T	⊙	T	F	T	T	T	F	F	F	T	⊙	⊙	T
F	T	⊙	T	F	T	⊙	T	F	F	F	T	⊙	F	⊙
F	T	⊙	T	F	T	F	T	F	F	F	T	⊙	F	F
F	T	F	T	T	T	T	T	F	F	T	T	F	F	T
F	T	F	T	T	⊙	⊙	T	F	F	T	T	F	F	⊙
F	T	F	T	T	F	F	T	F	F	T	T	F	F	F
F	T	F	T	⊙	T	T	T	F	F	⊙	T	F	F	T
F	T	F	T	⊙	T	⊙	T	F	F	⊙	T	F	F	⊙
F	T	F	T	⊙	⊙	F	T	F	F	⊙	T	F	F	F
F	T	F	T	F	T	T	T	F	F	F	T	F	F	T
F	T	F	T	F	T	⊙	T	F	F	F	T	F	F	⊙
F	T	F	T	F	T	F	T	F	F	F	T	F	F	F

Derivation Systems for 3-Valued Propositional Logic

Completeness of 3-valued Łukasiewicz's Logic

Rem. 4.50 Recall that in defn. 2.16 Fronhöfer abbreviated the formula $\sim(p_1 \supset p_2)$ as **f**, we also decided in Rem 1.76 to use $\sim_{\mathrm{BE}}$ to represent 'not' in Bochvar's External System $\mathbf{B}_{\mathbf{3}}^{\mathbf{E}}$ where $(A \supset \sim A)$ is abbreviated as $\sim_{\mathrm{BE}}A$. Recall also the truth table for $(A \supset \sim A)$ is

A	⊃	~	A
T	F	F	T
⊙	T	⊙	⊙
F	T	T	F

(p. 302)

Additional Derived Formulae

Ł$_3$D30 $\vdash (A \supset \sim_{\mathrm{BE}}A) \supset \sim_{\mathrm{BE}}A$

$Ł_3$**D31** $\sim_{\mathrm{BE}}\sim_{\mathrm{BE}}A \supset A$

$Ł_3$**D32** $(A \supset B) \supset (\sim B \supset \sim A)$

$Ł_3$**D33** $\sim(A \supset B) \supset \sim B$

$Ł_3$**D34** $A \supset (\sim B \supset \sim(A \supset B))$

$Ł_3$**D35** $\vdash \sim_{\mathrm{BE}}A \supset (\sim_{\mathrm{BE}}\sim B \supset (A \supset B))$ (p. 302)

Fronhöfer's derivations follow:

$Ł_3$**D30** $\vdash (A \supset \sim_{\mathrm{BE}}A) \supset \sim_{\mathrm{BE}}A$

1.	$(((A \supset \sim A) \supset A) \supset A) \supset ((A \supset (A \supset \sim A)) \supset (((A \supset \sim A) \supset A) \supset (A \supset \sim A)))$	$Ł_3 2$, $(A \supset \sim A) \supset A/P$, A/Q, $A \supset \sim A/R$
2.	$((A \supset \sim A) \supset A) \supset A$	$Ł_3 4$, A/P
3.	$(A \supset (A \supset \sim A)) \supset (((A \supset \sim A) \supset A) \supset (A \supset \sim A))$	1,2 M.P
4.	$\sim A \supset (A \supset \sim A)$	$Ł_3 1$, $\sim A/P$, A/Q
5.	$\sim A \supset \sim\sim(A \supset \sim A)$	4 D.N.
6.	$\sim(A \supset \sim A) \supset A$	5 CON
7.	$((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset ((\sim(A \supset \sim A) \supset A) \supset ((A \supset \sim A) \supset A))$	$Ł_3 2$, $A \supset \sim A/P$, $\sim(A \supset \sim A)/Q$, A/R
8.	$((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset ((A \supset \sim A) \supset A)$	6,7 GMP
9.	$((A \supset \sim A) \supset A) \supset ((A \supset (A \supset \sim A)) \supset (A \supset \sim A))$	3 TRAN
10.	$((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset (A \supset (A \supset \sim A)) \supset ((A \supset \sim A))$	8,9 GHS
11.	$(A \supset (A \supset \sim A)) \supset (((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset (A \supset \sim A))$	10 TRAN
12.	$(((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset (A \supset \sim A)) \supset (A \supset \sim A)$	$Ł_3 4$, $A \supset \sim A/P$
13.	$(A \supset (A \supset \sim A)) \supset (A \supset \sim A))$	11,12 GHS (p. 304)

$Ł_3$**D31** $\sim_{BE}\sim_{BE}A \supset A$

1.	$\sim(A \supset \sim A) \supset A$	$Ł_3$D8, A/P, $\sim A/Q$
2.	$((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset$ $((\sim(A \supset \sim A) \supset A) \supset$ $((A \supset \sim A) \supset A))$	$Ł_3$2, $A \supset \sim A/P$ $\sim(A \supset \sim A)/Q$, A/R
3.	$(\sim(A \supset \sim A) \supset A) \supset$ $(((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset$ $((A \supset \sim A) \supset A))$	2 TRAN
4.	$((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset$ $((A \supset \sim A) \supset A)$	3,1 M.P.
5.	$((A \supset \sim A) \supset A) \supset A$	$Ł_3$4, A/P
6.	$((A \supset \sim A) \supset \sim(A \supset \sim A)) \supset A$	4,5 H.S.

Note: $Ł_3$D31 does not hold in reverse. See truth table below.

A	$\supset$	$\sim_{BE}$	$\sim_{BE}$	A
T	T	T	F	T
$\odot$	$\odot$	F	T	$\odot$
F	T	F	T	F

(p. 304)

$Ł_3$**D32** $(A \supset B) \supset (\sim B \supset \sim A)$

1.	$(\sim\sim A \supset \sim\sim B) \supset (\sim B \supset \sim A)$	$Ł_3$3, $\sim A/P$, $\sim B/Q$
2.	$(A \supset \sim\sim B) \supset (\sim B \supset \sim A)$	1 D.N.
3.	$(A \supset B) \supset (\sim B \supset \sim A)$	2 D.N. (p. 306)

$Ł_3$**D33** $\sim(A \supset B) \supset \sim B$

1.	$B \supset (A \supset B)$	$Ł_3$1, B/P, A/Q
2.	$\sim\sim B \supset (A \supset B)$	1 D.N.
3.	$\sim\sim B \supset \sim\sim(A \supset B)$	2 D.N.

4. $\sim(A \supset B) \supset \sim B$ 3 CON (p. 306)

Ł$_3$D34 $A \supset (\sim B \supset \sim(A \supset B))$

1. $A \supset ((A \supset B) \supset B)$ Ł$_3$D6
2. $(\sim B \supset \sim(A \supset B)) \supset ((A \supset B) \supset B)$ Ł$_3$3, B/P, $A \supset B/Q$
3. $((A \supset B) \supset B) \supset (\sim B \supset \sim(A \supset B))$ Ł$_3$3D32, $A \supset B/A$, B/B
4. $A \supset (\sim B \supset \sim(A \supset B))$ 1,2,3 Sub. (p. 306)

Ł$_3$D35 $\vdash \sim_{\mathrm{BE}}A \supset (\sim_{\mathrm{BE}}\sim B \supset (A \supset B))$

Fronhöfer refers the reader to Wajsberg (1931, p. 269) for a derivation; however we cannot make sense of the historical notation. We have therefore again resorted to the "brute force" method of proof by means of a truth table. If we accept that the statement is tautology and that such a demonstration is as good as a derivation, we may insert the symbol '⊢' as an indication of proof.

$\sim_{\mathrm{BE}}$	A	⊃	($\sim_{\mathrm{BE}}$	~	B	⊃	(A	⊃	B))
F	T	T	T	F	T	T	T	T	T
F	T	T	T	⊙	⊙	⊙	T	⊙	⊙
F	T	T	F	T	F	T	T	F	F
T	⊙	T	T	F	T	T	⊙	T	T
T	⊙	T	T	⊙	⊙	T	⊙	T	⊙
T	⊙	T	F	T	F	T	⊙	⊙	F
T	F	T	T	F	T	T	F	T	T
T	F	T	T	⊙	⊙	T	F	T	⊙
T	F	T	F	T	F	T	F	T	F

Lem. 4.51 (a) If $\boldsymbol{S}$ is syntactically inconsistent, then $\boldsymbol{S} \vdash \mathbf{A}$ for every formula $\mathbf{A}$ in Ł$_3$.

(b) $\boldsymbol{S}$ is syntactically inconsistent if, and only if, $\boldsymbol{S} \vdash$ **f**.

(c) If $\boldsymbol{S} \cup \{\mathbf{A}\}$ is syntactically inconsistent, then $\boldsymbol{S} \vdash \sim_{\mathrm{BE}}\mathbf{A}$. (Recall $\sim_{\mathrm{BE}}\mathrm{A} \equiv \mathrm{A} \supset \sim\mathrm{A}$)

(d) If $\boldsymbol{S} \cup \{\sim_{\mathrm{BE}}\mathbf{A}\}$ is syntactically inconsistent, then $\boldsymbol{S} \vdash \mathbf{A}$. (p. 308)

Proof: (a) Suppose that $\boldsymbol{S} \vdash \boldsymbol{B}$ and $\boldsymbol{S} \vdash \sim\boldsymbol{B}$ for some formula $\mathbf{B}$ of Ł$_3$, then

1. B given
2. $\sim B$ given
3. $\sim B \supset (B \supset A)$ Ł$_3$D1, B/P, A/Q

4. $B \supset A$ 3,2 M.P.

5. A 4,1 M.P.

Therefore $\boldsymbol{S} \vdash \mathbf{A}$ for every formula $\mathbf{A}$ of Ł$_3$. (p. 308)

(b) On the one hand, $p_1 \supset p_1$ is an instance of Ł$_3$D4 ($P \supset P$)

By lemma 4.9(b) $\boldsymbol{S} \vdash p_1 \supset p_1$

By Ł$_3$D3 ($P \supset \sim\sim P$) we get $\boldsymbol{S} \vdash \sim\sim(p_1 \supset p_1)$

Therefore, $\boldsymbol{S} \vdash \sim \mathbf{f}$

Hence, if $\boldsymbol{S} \vdash \mathbf{f}$ as well, then $\boldsymbol{S}$ is syntactically inconsistent.

On the other hand however, if $\boldsymbol{S}$ is syntactically inconsistent then we get $\boldsymbol{S} \vdash \mathbf{f}$ as a special case of Lemma 4.51 with $\mathbf{A}$ for $\mathbf{f}$.

(c) Suppose $\boldsymbol{S} \cup \{\mathbf{A}\}$ is syntactically inconsistent, then

by Lemma 4.51(a) $\boldsymbol{S} \cup \{\mathbf{A}\} \vdash \sim\mathbf{A}$ for an arbitrary formula $\sim\mathbf{A}$.

Hence, $\boldsymbol{S} \vdash \mathbf{A} \supset (\mathbf{A} \supset \sim\mathbf{A})$ by Lemma 4.48 (Stutterer's Deduction Theorem).

I.e. $\boldsymbol{S} \vdash \boldsymbol{A} \supset \sim_{\mathrm{BE}}\mathbf{A}$.

Hence, $\boldsymbol{S} \vdash \sim_{\mathrm{BE}}\mathbf{A}$ by Ł$_3$D30: $(A \supset \sim_{\mathrm{BE}}A) \supset \sim_{\mathrm{BE}}A$ and Lemma 4.9(f) via M.P.

(d) By Lemma 4.51(c) above, we get,

$\boldsymbol{S} \vdash \sim_{\mathrm{BE}}\sim_{\mathrm{BE}}\mathbf{A} \qquad \equiv (\sim_{\mathrm{BE}}(\mathbf{A} \supset \sim\mathbf{A}) \equiv (\mathbf{A} \supset \sim\mathbf{A}) \supset \sim(\mathbf{A} \supset \sim\mathbf{A}))$

With Ł$_3$D31 ($\sim_{\mathrm{BE}}\sim_{\mathrm{BE}}\mathbf{A} \supset \mathbf{A}$) and Lemma 4.9(f), we get

$\boldsymbol{S} \vdash \mathbf{A}$ via M.P. (p. 308)

Rem. 4.52 It can be proved that, if a set of formulae $\boldsymbol{S}$ of Ł$_3$ is syntactically consistent, then $\boldsymbol{S}$ is semantically consistent as well. To do so Fronhöfer proceeds as follows:

- Begin with a two-valued precedent *i.e.* assume $\boldsymbol{S}$ to be syntactically consistent, then extend $\boldsymbol{S}$ into a superset $\boldsymbol{S}_\infty$ by iteratively adding formulae while preserving consistency.

- Then show that the formulae of $\boldsymbol{S}_\infty$, and hence of $\boldsymbol{S}$, evaluate to T under some suitably constructed interpretation. (p. 310)

Defn. 4.53 Given a syntactically consistent set of formulae $\boldsymbol{S}$ of Ł$_3$, we can construct a set of formulae $\boldsymbol{S}_\infty$ of Ł$_3$ as follows:

- Define $\boldsymbol{S}_0$ as $\boldsymbol{S}$.
- Assume that the formulae of Ł$_3$ can be alphabetically ordered so that $\mathbf{A}_i$ represents each such formula from i = 1 on alphabetically to the ith formula of Ł$_3$ and define:

$$\boldsymbol{S}_i = \begin{cases} \boldsymbol{S}_{i-1} \cup \{\mathbf{A}_i\} \\ \boldsymbol{S}_{i-1} \end{cases}$$

  if $\boldsymbol{S}_{i-1} \cup \{\mathbf{A}_i\}$ is syntactically consistent, above, and $\boldsymbol{S}_{i-1}$ otherwise.

- Then define $\boldsymbol{S}_\infty$ as $\bigcup_{i \in \mathbb{N}} \mathcal{S}_i$.

Lem. 4.54 (a) $\boldsymbol{S}_\infty$ is syntactically consistent.

(b) $\boldsymbol{S}_\infty$ is maximally consistent. (p. 310)

Proof: (a) We proceed by *reductio ad absurdum*:

Suppose that $\boldsymbol{S}_\infty$ were syntactically inconsistent. Then at least one finite subset $\boldsymbol{S}'$ of $\boldsymbol{S}_\infty$ would be syntactically inconsistent because inconsistency implies that **f** is derivable from $\boldsymbol{S}_\infty$ by Lemma 4.51(b) and **f** is derivable from a finite subset $\boldsymbol{S}'$ of $\boldsymbol{S}_\infty$ by Lemma 4.9(d).

But for a finite subset $\boldsymbol{S}'$, each formula within $\boldsymbol{S}'$ must be one of $\boldsymbol{S}_i$, and since these are ordered linearly by the relation $\subseteq$, $\boldsymbol{S}'$is a subset of the greatest of these $\boldsymbol{S}_i$'s selected via the formulae in $\boldsymbol{S}'$.

But, $\boldsymbol{S}_0$ is consistent by definition and each of $\boldsymbol{S}_1$, $\boldsymbol{S}_2$, *etc*. is consistent by construction; therefore our assumption leads to a contradiction.

(b) Suppose $\boldsymbol{S}_\infty \nvdash \mathbf{A}$, where $\mathbf{A}$ is alphabetically the ith formula of Ł$_3$.

Then by Lemma 4.9(e) $\mathbf{A} \notin \boldsymbol{S}_\infty$. Consequently, $\mathbf{A} \notin \boldsymbol{S}_i$ due to the construction of $\boldsymbol{S}_\infty$ and $\boldsymbol{S}_i$. Therefore $\boldsymbol{S}_{i-1} \cup \{\mathbf{A}\}$ is syntactically inconsistent, due to the construction of $\boldsymbol{S}_i$.

Hence Lemma 4.51(b) obtains, *i.e.* $\boldsymbol{S}_{i-1} \cup \{\mathbf{A}\} \vdash$ **f**, and therefore Lemma 4.9(a) also obtains, *i.e.* $\boldsymbol{S}_\infty \cup \{\mathbf{A}\} \vdash$ **f**. Finally by Lemma 4.51(b) again, the set $\boldsymbol{S}_\infty \cup \{\mathbf{A}\}$ must be syntactically inconsistent. (p. 312)

Defn. 4.55 Let $\mathbf{I}_\infty$ be the result of assigning the following to each propositional variable $\mathbf{P}$ of Ł$_3$:

- T if $\boldsymbol{S}_\infty \vdash \mathbf{P}$ (and hence by the syntactic consistency of $\boldsymbol{S}_\infty$, $\boldsymbol{S}_\infty \nvdash {\sim}\mathbf{P}$)

- F if $\boldsymbol{\mathcal{S}}_\infty \vdash \sim\mathbf{P}$ (and hence by the syntactic consistency of $\boldsymbol{\mathcal{S}}_\infty$, $\boldsymbol{\mathcal{S}}_\infty \nvdash \mathbf{P}$)
- otherwise $\odot$.

Lem. 4.56 For any formula $\mathbf{A}$ of Ł$_3$ the following obtains:

i. if $\boldsymbol{\mathcal{S}}_\infty \vdash \mathbf{A}$ (and hence $\boldsymbol{\mathcal{S}}_\infty \nvdash \sim\mathbf{A}$) then $\mathbf{A}^{\mathbf{I}_\infty} = \mathrm{T}$,

ii. if $\boldsymbol{\mathcal{S}}_\infty \vdash \sim\mathbf{A}$ (and hence $\boldsymbol{\mathcal{S}}_\infty \nvdash \mathbf{A}$) then $\mathbf{A}^{\mathbf{I}_\infty} = \mathrm{F}$,

iii. if neither of the above then $\mathbf{A}^{\mathbf{I}_\infty} = \odot$. (p. 312)

Proof: We proceed by mathematical induction on the length $\ell(\mathbf{A})$ of $\mathbf{A}$ from $k < n$ to n.

We define the length $\ell(\mathbf{A})$ of a formula $\mathbf{A}$ as:

- $\ell(\mathbf{A})$ of a propositional variable $= 1$
- $\ell(\sim\mathbf{A})$ of a negation of $\mathbf{A} = \ell(\mathbf{A}) + 1$
- $\ell(\mathbf{A} \supset \mathbf{B})$ of conditional $\mathbf{A} \supset \mathbf{B} = \ell(\mathbf{A}) + \ell(\mathbf{B}) + 1$

Base case: $\ell(\mathbf{A}) = 1$, hence $\mathbf{A}$ is a propositional variable. Then the assertion holds by the construction of $\mathbf{I}_\infty$ in Defn 4.55 above.

Hypothesis: Lemma 4.56 obtains for any formula $\mathbf{F} \mid \ell(\mathbf{F}) < \ell(\mathbf{A})$.

Induction step: $\ell(\mathbf{A}) > 1$.
Consider as case 1 and case 2 that:
$\mathbf{A}$ is either a negation $\sim\mathbf{B}$ or a conditional $\mathbf{B} \supset \mathbf{C}$.

For both cases we must consider each of the sub-cases i. - iii. above:
$\boldsymbol{\mathcal{S}}_\infty \vdash \mathbf{A}$, $\boldsymbol{\mathcal{S}}_\infty \vdash \sim\mathbf{A}$ or that neither of these two is the case.

For sub-cases i. - iii. above we have to show that:
$\mathbf{A}^{\mathbf{I}_\infty} = \mathrm{T}$, $\mathbf{A}^{\mathbf{I}_\infty} = \mathrm{F}$ and $\mathbf{A}^{\mathbf{I}_\infty} = \odot$ respectively. (p. 314)

Case 1: $\mathbf{A}$ is a negation $\sim\mathbf{B}$.

i. Suppose that $\boldsymbol{\mathcal{S}}_\infty \vdash \sim\mathbf{B}$, then $\boldsymbol{\mathcal{S}}_\infty \nvdash \mathbf{B}$ since $\boldsymbol{\mathcal{S}}_\infty$ is consistent. Hence by I.H. (ii) $\mathbf{B}^{\mathbf{I}_\infty} = \mathrm{F}$ and $[\sim\mathbf{B}]^{\mathbf{I}_\infty} = \mathrm{T}$.

ii. Suppose that $\mathcal{S}_\infty \vdash \sim\sim\mathbf{B}$, then by Ł$_3$D2 and Lemma 4.9(f), $\mathcal{S}_\infty \vdash \mathbf{B}$. Hence by I.H. $\mathbf{B}^{\mathbf{I}\infty} = \mathrm{T}$ and $[\sim\mathbf{B}]^{\mathbf{I}\infty} = \mathrm{F}$.

iii. Suppose that neither $\mathcal{S}_\infty \vdash \sim\mathbf{B}$ nor $\mathcal{S}_\infty \vdash \sim\sim\mathbf{B}$, then if $\mathbf{B}$ were provable from $\mathcal{S}_\infty$, then by Ł$_3$D3 and Lemma 4.9(f) $\sim\sim\mathbf{B}$ would be provable from $\mathcal{S}_\infty$, which is a contradiction. Hence neither $\mathcal{S}_\infty \vdash \mathbf{B}$ nor $\mathcal{S}_\infty \vdash \sim\mathbf{B}$. Therefore by I.H. $\mathbf{B}^{\mathbf{I}\infty} = \odot$ and $[\sim\mathbf{B}]^{\mathbf{I}\infty} = \odot$.

Case 2: $\mathbf{A}$ is a conditional $\mathbf{B} \supset \mathbf{C}$.

i. Suppose that $\mathcal{S}_\infty \vdash (\mathbf{B} \supset \mathbf{C})$, then:

If $\mathcal{S}_\infty \vdash \sim\mathbf{B}$, then $\mathbf{B}^{\mathbf{I}\infty} = \mathrm{F}$ by I.H.

If $\mathcal{S}_\infty \vdash \mathbf{C}$ then $\mathbf{C}^{\mathbf{I}\infty} = \mathrm{T}$ by I.H.

If $\mathcal{S}_\infty \vdash \mathbf{B}$ then $\mathcal{S}_\infty \vdash \mathbf{C}$ by Lemma 4.9(f) and hence again $\mathbf{C}^{\mathbf{I}\infty} = \mathrm{T}$

If $\mathcal{S}_\infty \vdash \sim\mathbf{C}$ then $\mathcal{S}_\infty \vdash \sim\mathbf{B}$ then by Lemma 4.9(a), Ł$_3$D32 and Lemma 4.9(f), $\mathbf{B}^{\mathbf{I}\infty} = \mathrm{F}$

Consequently,

If any one of $\mathbf{B}$, $\sim\mathbf{B}$, $\mathbf{C}$ or $\sim\mathbf{C}$ is provable from $\mathcal{S}_\infty$, then $\mathbf{B}^{\mathbf{I}\infty} = \mathrm{F}$ or $\mathbf{C}^{\mathbf{I}\infty} = \mathrm{T}$ and therefore $(\mathbf{B} \supset \mathbf{C})^{\mathbf{I}\infty} = \mathrm{T}$.

If neither $\mathbf{B}$, $\sim\mathbf{B}$, $\mathbf{C}$ or $\sim\mathbf{C}$ is provable from $\mathcal{S}_\infty$, then $\mathbf{B}^{\mathbf{I}\infty} = \mathbf{C}^{\mathbf{I}\infty} = \odot$, and by I.H. therefore $(\mathbf{B} \supset \mathbf{C})^{\mathbf{I}\infty} = \mathrm{T}$. (p. 316)

ii. Suppose that $\mathcal{S}_\infty \vdash \sim(\mathbf{B} \supset \mathbf{C})$, then:

from $\sim(\mathbf{B} \supset \mathbf{C})$ and Ł$_3$D8 $\mathbf{B}$ follows and with Ł$_3$D33 $\sim\mathbf{C}$ follows, and with Lemma 4.9(f) both $\mathcal{S}_\infty \vdash \mathbf{B}$ and $\mathcal{S}_\infty \vdash \sim\mathbf{C}$ follow. With the I.H. this implies that $\mathbf{B}^{\mathbf{I}\infty} = \mathrm{T}$ and $\mathbf{C}^{\mathbf{I}\infty} = \mathrm{F}$ and hence $(\mathbf{B} \supset \mathbf{C})^{\mathbf{I}\infty} = \mathrm{F}$. (p. 316)

iii. Suppose that neither $\mathcal{S}_\infty \vdash (\mathbf{B} \supset \mathbf{C})$ nor $\mathcal{S}_\infty \vdash \sim(\mathbf{B} \supset \mathbf{C})$, then neither $\sim\mathbf{B}$ nor $\mathbf{C}$ can be proved from $\mathcal{S}_\infty$ because:

with Ł$_3$D1, Lemma 4.9(a) and Lemma 4.9(f) would lead to the contradiction that $(\mathbf{B} \supset \mathbf{C})$ is provable from $\mathcal{S}_\infty$, and with Ł$_3$3 we would be able to prove that $\mathcal{S}_\infty \vdash (\mathbf{B} \supset \mathbf{C})$ from $\mathcal{S}_\infty \vdash \mathbf{C}$.

So $\mathbf{B}^{\mathbf{I}\infty} \neq \mathrm{F}$ and $\mathbf{C}^{\mathbf{I}\infty} \neq \mathrm{T}$, because:

in the case that $\mathcal{S}_\infty \nvdash \sim\mathbf{B}$ we would get either $\mathcal{S}_\infty \vdash \mathbf{B}$ and hence $\mathbf{B}^{\mathbf{I}\infty} = \mathrm{T}$ by I.H. or $\mathcal{S}_\infty \nvdash \mathbf{B}$ and hence $\mathbf{B}^{\mathbf{I}\infty} = \odot$ also by I.H. and analogously $\mathcal{S}_\infty \nvdash \mathbf{C}$. (p. 316)

Now suppose firstly that $\mathbf{B}^{\mathbf{I}_\infty} = \mathrm{T}$, then $\mathbf{C}^{\mathbf{I}_\infty} \neq \mathrm{F}$ because by I.H. $\sim\mathbf{C}$ would then be provable from $\boldsymbol{\mathcal{S}}_\infty$, and hence by Ł$_3$D34 and Lemma 4.9(f) so would $\sim(\mathbf{B} \supset \mathbf{C})$. Since, as shown above $\mathbf{C}^{\mathbf{I}_\infty} \neq \mathrm{T}$ either, $\mathbf{C}^{\mathbf{I}_\infty} = \odot$, and hence $(\mathbf{B} \supset \mathbf{C})^{\mathbf{I}_\infty} = \odot$.

Suppose next that $\mathbf{B}^{\mathbf{I}_\infty} = \odot$ and that we assume that $\mathbf{C}^{\mathbf{I}_\infty} = \odot$, then by I.H. neither $\mathbf{B}$ nor $\sim\mathbf{C}$ would be provable from $\boldsymbol{\mathcal{S}}_\infty$. But because $\boldsymbol{\mathcal{S}}_\infty$ is maximally consistent, both $\mathcal{S} \cup \{\mathbf{B}\}$ and $\mathcal{S} \cup \{\sim\mathbf{C}\}$ would be syntactically inconsistent. Hence by Lemma 4.51(c) both $\sim_{\mathrm{BE}}\mathbf{B}$ and $\sim_{\mathrm{BE}}\sim\mathbf{C}$ would be provable from $\boldsymbol{\mathcal{S}}_\infty$, and hence by Ł$_3$D35 and Lemma 4.9(f), so would $(\mathbf{B} \supset \mathbf{C})$ also.

Since $\mathbf{C}^{\mathbf{I}_\infty} \neq \mathrm{T}$, above, $\mathbf{C}^{\mathbf{I}_\infty} = \mathrm{F}$ and hence $(\mathbf{B} \supset \mathbf{C})^{\mathbf{I}_\infty} = \odot$.
(p. 318)

Lem. 4.57 If $\boldsymbol{\mathcal{S}}$ is syntactically consistent, then $\boldsymbol{\mathcal{S}}$ is semantically consistent.

Proof: Since every member of $\boldsymbol{\mathcal{S}}$ belongs to $\boldsymbol{\mathcal{S}}_\infty$, and hence is provable from $\boldsymbol{\mathcal{S}}_\infty$ by Lemma 4.9(e); therefore every member of $\boldsymbol{\mathcal{S}}$ will evaluate to T under $\mathbf{I}_\infty$. (p. 318)

Completeness Theorem

Thrm. 4.58 For every set of formulae $\boldsymbol{\mathcal{S}}$ of Ł$_3$ and every formula $\mathbf{A}$ of Ł$_3$, the following obtains:

- If $\boldsymbol{\mathcal{S}} \vDash \mathbf{A}$ then $\boldsymbol{\mathcal{S}} \vdash \mathbf{A}$ (Strong Completeness Theorem)
- If $\vDash \mathbf{A}$ then $\vdash \mathbf{A}$ (Weak Completeness Theorem)

(p. 320)

Proof: Suppose that $\boldsymbol{\mathcal{S}} \vDash \mathbf{A}$, then $\boldsymbol{\mathcal{S}} \cup \{\sim_{\mathrm{BE}}\mathbf{A}\}$ would be semantically inconsistent, because $\mathbf{A}$ would be true in every model of $\boldsymbol{\mathcal{S}}$, therefore $\sim\mathbf{A}$ would be false in every model of $\boldsymbol{\mathcal{S}}$, and so would $\sim_{\mathrm{BE}}\mathbf{A} = \mathbf{A} \supset \sim\mathbf{A}$.

Consequently, by Lemma 4.57 we get that $\boldsymbol{\mathcal{S}} \cup \{\sim_{\mathrm{BE}}\mathbf{A}\}$ is syntactically inconsistent, and hence by Lemma 4.51(d) that $\boldsymbol{\mathcal{S}} \vdash \mathbf{A}$.

For the special as of $\boldsymbol{\mathcal{S}}$ as $\emptyset$, we get the Weak Completeness Theorem as a consequence.

(p. 320)

Decidability

Lem. 4.59 The set of theorems of Ł$_3$**A** is decidable.

Proof: This is so because Ł$_3$**A** is both a sound and complete system of Ł$_3$, and the set of tautologies of Ł$_3$ is decidable by constructing truth tables. (p. 320)

Application of Derivation Systems for 3-Valued Propositional Logic: Independence of Axioms

Rem. 4.60 One of the first applications of many valued logic was for proofs of independence.

Consider the following axiom system $\mathcal{K}$ for classical propositional logic:

Ax. 1 $p_1 \supset (p_2 \supset p_1)$

Ax. 2 $((p_1 \supset p_2) \supset p_1) \supset p_1$

Ax. 3 $(p_1 \supset p_2) \supset ((p_2 \supset p_3) \supset (p_1 \supset p_3))$

Ax. 4 $(p_1 \bullet p_2) \supset p_1$

Ax. 5 $(p_1 \bullet p_2) \supset p_2$

Ax. 6 $(p_1 \supset p_2) \supset ((p_1 \supset p_3) \supset p_1 \supset p_2 \bullet p_3))$

Ax. 7 $p_1 \supset (p_1 \vee p_2)$

Ax. 8 $p_2 \supset (p_1 \vee p_2)$

Ax. 9 $(p_1 \supset p_3) \supset ((p_2 \supset p_3) \supset p_1 \vee p_2 \supset p_3))$

Ax. 10 $(p_1 \equiv p_2) \supset (p_1 \supset p_2)$

Ax. 11 $(p_1 \equiv p_2) \supset (p_2 \supset p_1)$

Ax. 12 $(p_1 \supset p_2) \supset ((p_2 \supset p_1) \supset p_1 \equiv p_2))$

Ax. 13 $(p_1 \supset p_2) \supset ({\sim}p_2 \supset {\sim}p_1)$

Ax. 14 $p_1 \supset {\sim\sim}p_1$

Ax. 15 ${\sim\sim}p_1 \supset p_1$ (p. 322)

Defn. 4.61 An axiom system K is **independent** iff for every axiom $A \in K$ is not derivable from the axiom set $K\backslash\{A\}$. It is then said that A is independent from the other axioms of K.

Lem. 4.62 Axiom Ax. 2 is independent of the other axioms of $\mathcal{K}$.

Proof: Consider the axiom system $\mathcal{K}$ from the point of view of $Ł_3$, then show that:

a) every axiom of $\mathcal{K}$, apart from Ax. 2, is a tautology of $Ł_3$
b) all the rules of our calculus (in our case just M.P.) lead from $Ł_3$ tautologies to an $Ł_3$ tautology, and that
c) Ax. 2 is not a tautology of $Ł_3$, thus:

a) can be shown by truth tables.

b) We know that M.P. is truth preserving for $Ł_3$.
c) $((P \supset Q) \supset P) \supset P$ is not a tautology of $Ł_3$. See below.

((P	⊃	Q)	⊃	P)	⊃	P
T	T	T	T	T	T	T
T	⊙	⊙	T	T	T	T
T	F	F	T	T	T	T
⊙	T	T	⊙	⊙	T	⊙
⊙	T	⊙	⊙	⊙	T	⊙
⊙	⊙	F	T	⊙	⊙	⊙
F	T	T	F	F	T	F
F	T	⊙	F	F	T	F
F	T	F	F	F	T	F

a), b) and c) above imply that every formula B, which is derivable from $\mathcal{K}\backslash\{\text{Ax. 2}\}$ must be a tautology in $Ł_3$. Consequently, $\mathcal{K}\backslash\{\text{Ax. 2}\} \nvdash$ Ax. 2 because Ax. 2 is not a tautology in $Ł_3$. (p. 324)

A Pavelka-Style Derivation System for Łukasiewicz Logic

General Principle and Motivation (Fronhöfer)

- In Pavelka-style systems we introduce constant names for truth-values and annotate formulae in derivations with truth-values (graded formulae).

- With this added expressive power we are be able to use derivations, not only to establish tautologousness and validity, but also quasi-tautologousness, quasi-validity, and degree-validity for 3-valued logics.

- We augment the language of $Ł_3$ with the following atomic formula: **t**, **f** and **n** with the truth values T, F and ⊙ respectively, on every truth-value assignment of these formulae.

- We assume the following ordering of truth-values: $\mathrm{F} < \odot < \mathrm{T}$

- This enables us to draw partially true consequences from partially true premises.

(p. 326)

Rem. 4.62 Given at truth-value constant **v** with value v:

- If true, the formula **v ⊃ P** can be understood as: **P** has at least the value v.

t	**P**	**T ⊃ P**
T	T	T
T	⊙	⊙
T	F	F

n	**P**	**n ⊃ P**
⊙	T	T
⊙	⊙	T
⊙	F	⊙

f	**P**	**f ⊃ P**
F	T	T
F	⊙	T
F	F	T

- If true, the formula **P ⊃ v** can be understood as: **P** has at most the value v, or **P** is no truer than v.

t	**P**	**P ⊃ t**
T	T	T
T	⊙	T
T	F	T

n	**P**	**P ⊃ n**
⊙	T	⊙
⊙	⊙	T
⊙	F	T

f	**P**	**P ⊃ f**
F	T	F
F	⊙	⊙
F	F	T

- Together, **v ≡ P** can be understood as: **P** has exactly the truth value v.

Defn. 4.63 A pair [**P**, v], where **P** is any formula and v is one of the three truth values T, F or ⊙ is known as a **graded formula**.

Rem 4.64 Intuitively, a graded formula expresses the following: the value v in the graded formula [**P**, v] indicates that the formula **P** has at least the value v, *i.e.* that **P** has at least, but not necessarily exactly, the value v. Fronhöfer's proof to follow in Lemma 4.81. (p. 328)

Rem 4.65 Producing a new axiomatic system Ł$_3$**PA** (for Ł$_3$ Pavelka-style axiomatic system) involves:

- adding axioms and a new rule **TCI** involving the truth-value constants to Ł$_3$**A**, *i.e.* every axiom is graded with the value T, and each rule specifies the grades involved in its application.
- Ł$_3$**PA** takes Łukasiewicz's negation and conditional as primitive connectives and consists of the axioms given in Definition 4.66 below, where Ł$_3$**P1** to Ł$_3$**P4** are graded versions of Ł$_3$**A**. (p. 330)

Defn. 4.66 Axiom Schema of Ł$_3$**PA**:

Ł$_3$**P1** [**P** ⊃ (**Q** ⊃ **P**), T]

Ł$_3$**P2** [(**P** ⊃ **Q**) ⊃ ((**Q** ⊃ **R**) ⊃ (**P** ⊃ **R**)), T]

Ł$_3$**P3** [(~**P** ⊃ ~**Q**) ⊃ (**Q** ⊃ **P**), T]

Ł$_3$**P4** [((**P** ⊃ ~**P**) ⊃ **P**) ⊃ **P**, T]

Ł$_3$**P5.1.1** [(**t** ⊃ **t**) ⊃ **t**, T]

Ł$_3$**P5.1.2** [**t** ⊃ (**t** ⊃ **t**), T]

Ł$_3$**P5.2.1** [(**t** ⊃ **n**) ⊃ **n**, T]

Ł$_3$**P5.2.2** [**n** ⊃ (**t** ⊃ **n**), T]

Ł$_3$**P5.3.1** [(**t** ⊃ **f**) ⊃ **f**, T]

Ł$_3$**P5.8.1** [(**f** ⊃ **n**) ⊃ **t**, T]

Ł$_3$**P5.8.2** [**t** ⊃ (**f** ⊃ **n**), T]

Ł$_3$**P5.9.1** [(**f** ⊃ **f**) ⊃ **t**, T]

Ł$_3$**P5.9.2** [**t** ⊃ (**f** ⊃ **f**), T]

Ł$_3$**P6.1.1** [~**t** ⊃ **f**, T]

Ł$_3$**P5.3.2** [**f** ⊃ (**t** ⊃ **f**), T]	Ł$_3$**P6.1.2** [**f** ⊃ ~**t**, T]
Ł$_3$**P5.4.1** [(**n** ⊃ **t**) ⊃ **t**, T]	Ł$_3$**P6.2.1** [~**n** ⊃ **n**, T]
Ł$_3$**P5.4.2** [**t** ⊃ (**n** ⊃ **t**), T]	Ł$_3$**P6.2.2** [**n** ⊃ ~**n**, T]
Ł$_3$**P5.5.1** [(**n** ⊃ **n**) ⊃ **t**, T]	Ł$_3$**P6.3.1** [~**f** ⊃ **t**, T]
Ł$_3$**P5.5.2** [**t** ⊃ (**n** ⊃ **n**), T]	Ł$_3$**P6.3.2** [**t** ⊃ ~**f**, T]
Ł$_3$**P5.6.1** [(**n** ⊃ **f**) ⊃ **n**, T]	Ł$_3$**P7.1** [**t**, T]
Ł$_3$**P5.6.2** [**n** ⊃ (**n** ⊃ **f**), T]	Ł$_3$**P7.2** [**n**, ⊙]
Ł$_3$**P5.7.1** [(**f** ⊃ **t**) ⊃ **t**, T]	Ł$_3$**P7.3** [**f**, F]
Ł$_3$**P5.7.2** [**t** ⊃ (**f** ⊃ **t**), T]	

(p. 330)

Defn. 4.67 There are two Graded Rules of Inference of Ł$_3$**PA**:

M.P. From [**P**, v_1] and [**P** ⊃ **Q**, v_2], infer [**Q**, v_3], where v_3 is defined in terms of v_1 and v_2 as specified in the following truth tables:

v_1	v_2	v_3
T	T	T
T	⊙	⊙
T	F	F

v_1	v_2	v_3
⊙	T	⊙
⊙	⊙	F
⊙	F	F

v_1	v_2	v_3
F	T	F
F	⊙	F
F	F	F

TCI (Truth-value Constant Introduction): from [**P**, v] infer [**v** ⊃ **P**, T] where **v** is the truth constant having the truth value v. (p. 332)

Motivation for the Pavelka Rules of Inference

Rem 4.68 The **TCI** allows us to move from a graded formula [**P**, v] to a formula **v** ⊃ **P** that states what the graded value is or where the graded value is integrated into the formula. (*cf.* 4.62)

Lem. 4.69 **CTCI (Converse of TCI)** We can move from a graded formula [**v** ⊃ **P**, T], where **v** is the name of the truth value v, to the graded formula [**P**, v].

Proof: This follows from the three axiom schemata Ł$_3$**P7.1** to Ł$_3$**P7.3** and M.P.

1. [**n** ⊃ **P**, T]	given	1. [**t** ⊃ **P**, T]	given	1. [**f** ⊃ **P**, T]	given
2. [**n**, ⊙]	Ł$_3$**P7.2**	2. [**t**, T]	Ł$_3$**P7.1**	2. [**f**, F]	Ł$_3$**P7.3**
3. [**P**, ⊙]	1,2 M.P.	3. [**P**,T]	1,2 M.P.	3. [**P**,F]	1,2. M.P.

Rem. 4.70 The with M.P. is to associate the derived formula **Q** with the least truth value that it could have, on the basis of the least values assigned to **P** ⊃ **Q** and **P**, thus:

P	Q
T	T
T	⊙
T	F
⊙	T
⊙	⊙
⊙	F
F	T
F	⊙
F	F

P = v_1	$(P \supset Q) = v_2$	Q?	M.P. = v_3
T	T	T	T
T	⊙	⊙	⊙
T	F	F	F
⊙	T	{T,⊙}	⊙
⊙	T	F	F
⊙	⊙	—	F
F	T	{T,⊙, F}	F
F	T	—	F
F	T	—	F

(p. 334)

Rem. 4.71 Rewriting the three tables from defn 4.67 above as a usual truth function table for v_3, we obtain:

$v_1 \backslash v_2$	T	⊙	F
T	T	⊙	F
⊙	⊙	F	F
F	F	F	F

Note that the table differs from the Łukasiewicz conjunction • for the combination ⊙ and ⊙ (magenta) but does coincides with the Łukasiewicz bold connective **&**.

(p. 333)

An Alternative Approach

Fronhöfer provides an alternative approach to a Pavelka-Style Derivation System for Łukasiewicz Logic, after Hájek (1998). We do not think such an alternative is necessary, especially since it is neither simpler nor more elegant. Nevertheless, we defer to Fronhöfer in including it here.

We keep the axioms of defn. 4.66, but replace defn 4.67 with the following:

Defn 4.72 [**P**, v] is shorthand for the ungraded formula **v** ⊃ **P**, where **v** is the name of the constant for the truth-value v.

T-Graded M.P. says that: From [**P** ⊃ **Q**, T] and [**P**, T], infer [**Q**, T].

This definition has several implications:

a) We may switch freely between [**P**, v] and **v** ⊃ **P**
b) The formula (**t** ⊃ (**t** ⊃ **F**)) ⊃ (**t** ⊃ **F**) obtains in Ł$_3$, as shown by the truth-table, over page:

(**t**	⊃	(**t**	⊃	**F**))	⊃	(**t**	⊃	**F**)
T	T	T	T	T	T	T	T	T
T	⊙	T	⊙	⊙	T	T	⊙	⊙
T	F	T	F	F	T	T	F	F

or is derivable with **t** as $\mathbf{p}_1 \supset \mathbf{p}_1$ as follows:

1. (**t** ⊃ (**t** ⊃ **F**)) ⊃ (**t** ⊃ (**t** ⊃ **F**)) Ł$_3$D4, **t** ⊃ (**t** ⊃ **F**)/**P**
2. **t** (= $\mathbf{p}_1 \supset \mathbf{p}_1$) Ł$_3$D4, $\mathbf{p}_1/\mathbf{P}$
3. (**t** ⊃ (**t** ⊃ **F**)) ⊃ (**t** ⊃ **F**) 1,2 GMP (p. 336)

c) This justifies M.P. on formulae graded with T, because (**t** ⊃ (**t** ⊃ **Q**)) may be derived from **t** ⊃ (**P** ⊃ **Q**) and **t** ⊃ **P** thus,

1. (**t** ⊃ (**t** ⊃ **Q**)) ⊃ (**t** ⊃ **Q**) tautology b) above
2. **t** ⊃ (**P** ⊃ **Q**) given
3. **t** ⊃ **P** given
4. **P** ⊃ (**t** ⊃ **Q**) 2 TRAN
5. **t** ⊃ (**t** ⊃ **Q**) 4,3 H.S.
6. **t** ⊃ **Q** 1,5 M.P.

d) (i.) From **t** ⊃ **G** and **t** we get **G** via M.P.

(ii.) From **G** we get **t** ⊃ **G** via Axiom Ł$_3$1 and M.P.

Consequently, we can move from **G** to **t** ⊃ **G** and back again, and more generally:

[**P**, v] ≡ **v** ⊃ **P** shorthand

[**P**, v] ≡ **t** ⊃ (**v** ⊃ **P**) Ł$_3$ derivations (i.) & (ii.) above, and

[**P**, v] ≡ [**v** ⊃ **P**, T] shorthand

Consequences:

- this allows us to derive M.P. with arbitrary graded formulae, below and
- we have Lemma 4.69 and **TCI** by definition. (p. 336)

Derivation of fully graded M.P.

1.	$[\mathbf{P}, v_1]$	given
2.	$[\mathbf{P} \supset \mathbf{Q}, v_2]$	given
3.	$[\mathbf{v}_1 \supset \mathbf{P}, \mathrm{T}]$	1 shorthand and d) above
4.	$[\mathbf{v}_2 \supset (\mathbf{P} \supset \mathbf{Q}), \mathrm{T}]$	2 shorthand and d) above
5.	$[((\mathbf{v}_1 \supset \mathbf{P}) \supset ((\mathbf{v}_2 \supset (\mathbf{P} \supset \mathbf{Q})) \supset$ $((\mathbf{v}_1 \mathbin{\&} \mathbf{v}_2) \supset (\mathbf{P} \mathbin{\&} (\mathbf{P} \supset \mathbf{Q}))))), \mathrm{T}]$	$Ł_3$D16, $\mathbf{v}_1/\mathbf{P}$, $\mathbf{v}_2/\mathbf{R}$ $\mathbf{P}/\mathbf{Q}$, $(\mathbf{P} \supset \mathbf{Q})/\mathbf{S}$
6.	$[(\mathbf{v}_2 \supset (\mathbf{P} \supset \mathbf{Q})) \supset ((\mathbf{v}_1 \mathbin{\&} \mathbf{v}_2) \supset (\mathbf{P} \mathbin{\&} (\mathbf{P} \supset \mathbf{Q}))), \mathrm{T}]$	5,3 M.P.
7.	$[(\mathbf{v}_1 \mathbin{\&} \mathbf{v}_2) \supset (\mathbf{P} \mathbin{\&} (\mathbf{P} \supset \mathbf{Q})), \mathrm{T}]$	6,4 M.P.
8.	$[(\mathbf{P} \mathbin{\&} (\mathbf{P} \supset \mathbf{Q})) \supset \mathbf{Q}, \mathrm{T}]$	$Ł_3$D15
9.	$[(\mathbf{v}_1 \mathbin{\&} \mathbf{v}_2) \supset \mathbf{Q}, \mathrm{T}]$	7,8 H.S.
10.	$(\mathbf{v}_1 \mathbin{\&} \mathbf{v}_2) \supset \mathbf{v}$	see below
11.	$\mathbf{v} \supset (\mathbf{v}_1 \mathbin{\&} \mathbf{v}_2)$	see below
12.	$[\mathbf{v} \supset \mathbf{Q}, \mathrm{T}]$	10, 11 Sub.
13.	$[\mathbf{Q}, \mathbf{\&}^* (v_1, v_2)]$	12 shorthand

According to Fronhöfer, for each pair of truth constants $\mathbf{v}_1$ and $\mathbf{v}_2$ there exists a unique truth constant **v** whose value is $\mathbf{\&}^*(v_1, v_2)$ where $\mathbf{\&}^*$ is the interpretation of the connective **&**, and we can derive the theorems $(\mathbf{v}_1 \mathbin{\&} \mathbf{v}_2) \supset \mathbf{v}$ and $\mathbf{v} \supset (\mathbf{v}_1 \mathbin{\&} \mathbf{v}_2)$. (p. 338)

Rem 4.73 Note that all of the theorems derived for $Ł_3$**A** above can be derived here too because:

- The axioms of $Ł_3$**A** are included in $Ł_3$**PA** and their graded values will all be T, because $Ł_3$**PA**'s axioms are graded with T, and
- when M.P. is applied to formulae graded with T it produces another formula graded with T.

Therefore, every formula in the justification for a derived axiom schema will be graded with the value T.

Fronhöfer prefixes the derived axiom numbers with $Ł_3$**P** rather than simply $Ł_3$ to emphasise that they are part of the Pavelka-style system. For the same reasons above, all of the rules derived for $Ł_3$**A** can be used in $Ł_3$**PA** to derive formulae graded with T that are themselves graded with T. However Fronhöfer does also derive fully graded versions of these rules.

We now proceed to derive:

- rules for dealing with gradings, and
- graded versions of our old rules of inference. (p. 338)

Formula: If **P** has the value T, then ~**P** has the value F, in other words:

- saying that **P** has the value T, is equivalent to saying that it has at least the value T, symbolised as **t** ⊃ **P**, and
- saying that ~**P** has the value F, is equivalent to saying that ~**P** has at most the value F, symbolised by ~**P** ⊃ **f**.

Derivation:

1.	[(**t** ⊃ **P**) ⊃ (**t** ⊃ **P**), T]	$Ł_3$**PD**4, **t** ⊃ **P/P**
2.	[(**t** ⊃ **P**) ⊃ (~**P** ⊃ ~**t**), T]	1 GCon.
3.	[~**t** ⊃ **f**, T]	$Ł_3$**P6.1.1**
4.	[(**t** ⊃ **P**) ⊃ (~**P** ⊃ **f**), T]	2,3 GHS

Formula: If **P** has the value F, then **P** ⊃ **Q** has the value T

Derivation:

1.	[**t** ⊃ ~**f**, T]	$Ł_3$**P6.3.2**
2.	[(**t** ⊃ ~**f**) ⊃ ((~**f** ⊃ ~**P**) ⊃ (**t** ⊃ ~**P**)), T]	$Ł_3$**P3**, **t/P**, ~**f/Q**, ~**P/R**
3.	[(~**f** ⊃ ~**P**) ⊃ (**t** ⊃ ~**P**), T]	2,1 M.P.
4.	[(**P** ⊃ **f**) ⊃ (**t** ⊃ ~**P**), T]	3 GCon.
5.	[~**P** ⊃ (**P** ⊃ **Q**), T]	$Ł_3$**PD1**, **P/P**, **Q/Q**
6.	[(**P** ⊃ **f**) ⊃ (**t** ⊃ (**P** ⊃ **Q**)), T]	4,5 GHS

Formula: if **P** has at least the value T, then it has at least the value ⊙

Derivation:

1.	[**t**, T]	$Ł_3$**P7.1**

2. $[\mathbf{t} \supset (\mathbf{n} \supset \mathbf{t}), \mathrm{T}]$ Ł$_3$**P5.4.2**

3. $[\mathbf{n} \supset \mathbf{t}, \mathrm{T}]$ 2,1 M.P.

4. $[(\mathbf{n} \supset \mathbf{t}) \supset ((\mathbf{t} \supset \mathbf{P}) \supset (\mathbf{n} \supset \mathbf{P})), \mathrm{T}]$ Ł$_3$**P2**, **n**/**P**, **t**/**Q**, **P**/**R**

5. $[(\mathbf{t} \supset \mathbf{P}) \supset (\mathbf{n} \supset \mathbf{P}), \mathrm{T}]$ 4,3 M.P (p. 340)

In what follows Fronhöfer demonstrates some derivations containing formulae graded with values other than T.

We can add graded assumptions to derivations, *e.g.* to derive $[\sim\mathbf{B}, \odot]$ from $[\mathbf{B} \supset \mathbf{C}, \mathrm{T}]$ and $[\sim\mathbf{C}, \odot]$:

1. $[\mathbf{B} \supset \mathbf{C}, \mathrm{T}]$ CPA

2. $[\sim\mathbf{C}, \odot]$ CPA

3. $[\sim\mathbf{C}, \supset \sim\mathbf{B}, \mathrm{T}]$ 1 GCon.

4. $[\sim\mathbf{B}, \odot]$ 3,2 M.P.

I.e. if $\sim$**C** has at least the value $\odot$, and **B** $\supset$ **C** is true, then $\sim$**B** has at least the value $\odot$. Note that Fronhöfer has applied the derived rule GCon. only to a formula graded with T. Then **TCI** will then allow us to derive **n** $\supset \sim$**B** graded with T.

5. $[\mathbf{n} \supset \sim\mathbf{B}, \mathrm{T}]$ 4 TCI (p. 342)

***E.g.* 4.74** The quasi-tautologousness of $A \mathrm{~v} \sim A$ can be expressed by the graded formula $[(A \mathrm{~v} \sim A), \odot]$ or respectively $[((A \supset \sim A) \supset \sim A), \odot]$ which can be derived as:

1. $[\sim A \supset ((A \supset \sim A) \supset \sim A), \mathrm{T}]$ Ł$_3$**P1**, $\sim A/P$, $A \supset \sim A/Q$

2. $[(\mathbf{n} \supset \sim A) \supset$ Ł$_3$**P2**, **n**/P, $\sim A/Q$
   $((\sim A \supset ((A \supset \sim A) \supset \sim A)) \supset$ $(A \supset \sim A) \supset \sim A/R$
   $(\mathbf{n} \supset ((A \supset \sim A) \supset \sim A))), \mathrm{T}]$

3. $[(\mathbf{n} \supset \sim A) \supset (\mathbf{n} \supset ((A \supset \sim A) \supset \sim A)), \mathrm{T}]$ 1,2 GMP

4. $[A \supset ((A \supset \sim A) \supset \sim A), \mathrm{T}]$ Ł$_3$**PD6**, A/P, $\sim A/Q$

5. $[(\mathbf{n} \supset A) \supset ((A \supset ((A \supset \sim A) \supset \sim A)) \supset$ Ł$_3$**P2**, **n**/P, $\sim A/Q$,
   $(\mathbf{n} \supset ((A \supset \sim A) \supset \sim A))), \mathrm{T}]$ $(A \supset \sim A) \supset \sim A/R$

6. $[(\mathbf{n} \supset A) \supset (\mathbf{n} \supset ((A \supset \sim A) \supset \sim A)), \mathrm{T}]$ 4,5 GMP

7. $[(\mathbf{n} \supset A) \vee (A \supset \mathbf{n}), \mathrm{T}]$ Ł$_3$**PD10**, $\mathbf{n}/P$, A/Q

8. $[(\mathbf{n} \supset A) \vee (\sim\mathbf{n} \supset \sim A), \mathrm{T}]$ 7 GCon.

9. $[\sim\mathbf{n} \supset \mathbf{n}, \mathrm{T}]$ Ł$_3$**P6.2.1**

10. $[\mathbf{n} \supset \sim\mathbf{n}, \mathrm{T}]$ Ł$_3$**P6.2.2**

11. $[(\mathbf{n} \supset A) \vee (\mathbf{n} \supset \sim A), \mathrm{T}]$ 8,9,10 Sub.

12. $[\mathbf{n} \supset ((A \supset \sim A) \supset \sim A), \mathrm{T}]$ 3,6,11 DE

13. $[\mathbf{n}, \odot]$ Ł$_3$**P7.2**

14. $[((A \supset \sim A) \supset \sim A), \odot]$ 12,13 M.P (p. 342)

According to Fronhöfer:

> Whenever we derive a graded formula without making any assumptions, we may regard the graded formula as a derived axiom. Thus, we can have derived axioms with values other than T; for example, we have just justified
>
> Ł$_3$**PD12**: $[P \vee \sim P, \odot]$
>
> This in turn allows derivations like

1. $[(P \vee \sim P) \supset Q, \mathrm{T}]$ CPA

2. $[P \vee \sim P, \odot]$ Ł$_3$**PD12**, P/P

3. $[Q, \odot]$ 1,2 M.P.

Rem. 4.75 Whereas in classical logic the truth of $(P \vee \sim P) \supset Q$ would make Q true, in Ł$_3$ however, the most we could say is that $(P \vee \sim P) \supset Q$ would make Q have at least the value $\odot$ as reflected in the derivation above. (p. 344)

Defn. 4.76 A formula **P** is a is **a theorem to degree v** in Ł$_3$**PA** if

- there is a proof (*i.o.w.* a derivation without assumptions) of $[\mathbf{P}, v]$ and
- there no proof of $[\mathbf{P}, w]$ with $w > v$.

The second condition is required to cover instances where we may have two proofs that give different values to a formula **P**.

Example: We can derive $[(P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P), \mathrm{T}]$ as follows:

1. $[(P \supset {\sim}{\sim}Q) \supset (P \supset {\sim}{\sim}Q), \mathrm{T}]$ — $Ł_3$**PD4**, $(P \supset {\sim}{\sim}Q)/P$
2. $[(P \supset {\sim}{\sim}Q) \supset (P \supset Q), \mathrm{T}]$ — 1 D.N.
3. $[(P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P), \mathrm{T}]$ — 2 GCon.

This establishes that the formula $[(P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P)$ is a theorem to degree T, since there is no truth valve greater than T. (p. 344)

Fronhöfer however brings our attention to the following derivation:

1. $[\mathbf{t}, \mathrm{T}]$ — $Ł_3$**P7.1**
2. $[\mathbf{t} \supset ({\sim}((P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P)) \supset \mathbf{t}), \mathrm{T}]$ — $Ł_3$**P1**, $\mathbf{t}/P$, ${\sim}((P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P))/Q$
3. $[{\sim}((P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P)) \supset \mathbf{t}, \mathrm{T}]$ — 1,2 M.P.
4. $[{\sim}\mathbf{t} \supset {\sim}{\sim}((P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P)), \mathrm{T}]$ — 3 GCon.
5. $[{\sim}\mathbf{t} \supset ((P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P)), \mathrm{T}]$ — 4 D.N.
6. $[\mathbf{f} \supset {\sim}\mathbf{t}, \mathrm{T}]$ — $Ł_3$**P6.1.2**
7. $[\mathbf{f} \supset ((P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P)), \mathrm{T}]$ — 5,6 H.S.
8. $[\mathbf{f}, \mathrm{F}]$ — $Ł_3$**P7.3**
9. $[(P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P), \mathrm{F}]$ — 7,8 M.P.

Rem. 4.77 The above derivation establishes that $(P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P)$ has at least the value F; however we do not want to leave it at this, especially since it is a tautology of $Ł_3$. So we need to consider that there may be (as shown previously) other derivations that grade the formula $(P \supset {\sim}{\sim}Q) \supset ({\sim}Q \supset {\sim}P)$ with T. (p. 346)

Note: An update to this study unit will be published in the coming weeks.

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