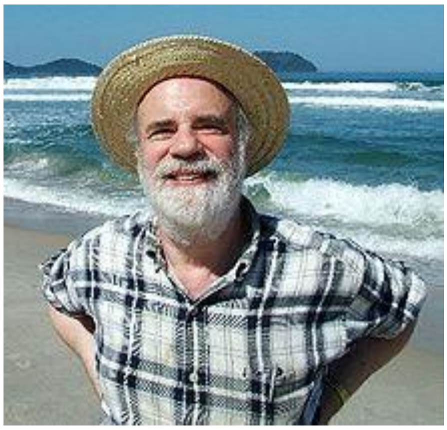
Critical Reasoning 23 – Introduction to Modal Logic

Modal logic is an extension of propositional and predicate logic to include operators expressing modality (from the Latin '*modus*' for 'manner', or 'way'.) Whereas a proposition P might true or false, the introduction of a modal operator qualifies *how* such a statement might be true or false. Compare the propositions:

It will rain tomorrow.

Possibly, it will rain tomorrow.



Saul Kripke (1940 -) *at age 77: American Philosopher and Logician and Cofounder of Kripke Semantics*

Clearly both propositions can be true or false; however the latter expresses 'the way' in which we might intend its truth or falsity. Similarly, with the propositions:

A triangle is a three sided figure.

Necessarily, a triangle is a three-sided figure.

Both of the above propositions express a truth; however the latter is much stronger – a triangle does not simply happen to have three sides, it *has to have* or *can only* have three sides. Possibility and necessity, which are modalities involving truth, hence also called **alethic modalities**, are defined in terms of each other using negation. Using the symbols '◊' for 'possible' and '□' for necessary we define: **Defn. 1**

$\Diamond P \equiv {\sim}\Box{\sim}P$ Read: "It is possible that P if, and only if, it is not necessary that not P" and

$\Box P \equiv {\sim}\Diamond{\sim}P$ Read: "It is necessary that P if, and only if, it is not possible that not P".

Contemporary modal logic includes other modalities such as **deontic modalities**,

O It is obligatory that…

P It is permitted that…

F It is forbidden that…

temporal modalities,

G It will always be the case that…

F It will be the case that…

H It has always been the case that…

P It was the case that…

doxastic modalities and **epistemic modalities** respectively such as

Bx x believes that... and It is known that P.

The following discussion follows the layout of McCance (2008) but also draws on the lecture notes of Zalta (1995) as well as the entry on 'Modal Logic' from the Stanford Encyclopedia of Philosophy. Wherever possible we have tried to harmonise the notation with that of Copi in pervious study units.

The Language of Modal Logic

The language of modal logic begins with the familiar set of symbols used for the propositional calculus, namely

- A countably finite set of symbols, such as A, B, C, ... , P_1, P_2, P_3, ... , known as 'propositional variables';
- The logical connectives •, v, ~, and ⊃;
- Parentheses ( ), [ ], { } and so on; to which are added the two unary modal operators;
- ◊ and □, pronounced 'diamond' and 'box'.

The next step is to define which combinations of these symbols count as a 'well-formed formulae' or '*wffs*' for short. According to the definition provided by McCance, a formula is a *wff* if it meets one of the following criteria: **Defn. 2.1**

- Any propositional variable P on its own is a *wff*.
- If α is a *wff,* then so are $(\sim\alpha)$, $(\Diamond\ \alpha)$ and $(\Box\alpha)$.
- If α and β are both *wffs*, then so are $(\alpha \vee \beta)$, $(\alpha \bullet \beta)$ and $(\alpha \supset \beta)$.

Note that this definition is **recursive**, defining first a 'base case' of 'atomic formulae' and then going on to define more complex formulae in terms of *wffs* already known to us. This provides us with a means for future proofs where we first prove something about an isolated propositional variable and then go on to show that it holds true for the second and third criteria above. This process is known as **induction on the complexity of a formula**, which McCance uses in some proofs of soundness and completeness below. (p. 3)

We have already seen in Critical Reasoning 05 how to use truth tables to check for tautologies of the propositional calculus. Recall that for tautologies, all the truth values under the major connective will be T's. Hitherto we have reserved the term 'valid' to be a property of arguments; however McCance uses the term '**valid formulae**' to denote formulae that are tautologies. According to Kripke's **many-world semantics** interpretation of modal logic, 'necessarily true' is given the reading, 'true in all possible worlds'. The truth value of propositional formulae is simply determined by the state of the world in question. $\Box P$ however is defined to be true in a world wherever P is true in all 'accessible' worlds. How we define **accessible** is relative to the modality in question but 'conceivable' is a common definition for the alethic modalities of necessity and possibility. Thus if P is true in all conceivable worlds then $\Box P$ is true, *i.e.* P is necessarily true. Similarly if P is true in at least one accessible world then $\Diamond\ P$ is true *i.e*. since it is true somewhere, it must not be impossible. (p. 3 - 4)

Consider the following example of a doxastic modality $\Box P$ to mean 'x believes P', where x is a person. The possible 'worlds' in this case are not 'worlds' at all, but anything capable of believing a proposition, such as persons, ideologies or institutions. Accessibility in this case may be interpreted as 'trusts' so that if x trusts something, then that thing is accessible to x. Thus, if Pythagoreans trust mathematics, then mathematics is accessible to them. In particular, if Russel is a 'node' in the network of persons, ideologies or institutions and Russell trusts only mathematics, logic and atheism, then $\Box P$ is true of Russell, if and only if, P is true for mathematics, logic and atheism. If however P is not true of any of these then Russell does not believe P. Suppose however that only atheism believes that P, then $\Diamond P$ is true for Russell. According to McCance, "He can see how one would believe P, but is not fully convinced." Finally, whether or not P is true in homeopathy does not affect Russel's beliefs because he doesn't trust it. Homeopathy is not accessible to him via the relation of 'trusts'. (p. 4)

With these examples in mind McCance defines the semantics of the above symbols.

Defn. 2.2 Let W be a non-empty set of what we may call 'possible worlds'. Let R be a binary relation from W onto W, which we may call an **accessibility relation**. Then $\langle W, R \rangle$ form a **frame**.

Defn. 2.3 If $\langle W, R \rangle$ is our frame, let $\Vdash$ (read 'forces') be a binary relation between W and the set of all *wffs*, also let $\Gamma \in W$. We assume that $\Vdash$ obeys the following rules:

- For all propositional variables P, either $\Gamma \Vdash P$ or $\Gamma \Vdash \sim P$.
- If α is a *wff*, then $\Gamma \Vdash \sim\alpha$ if, and only if, $\Gamma \nVdash \alpha$. (Read $\nVdash$ 'does not force.)
- If α and β are *wffs*, then $\Gamma \Vdash (\alpha \vee \beta)$ if, and only if, $\Gamma \Vdash \alpha$ or $\Gamma \Vdash \beta$.
- If α and β are *wffs*, then $\Gamma \Vdash (\alpha \bullet \beta)$ if, and only if, $\Gamma \Vdash \alpha$ and $\Gamma \Vdash \beta$.
- If α and β are *wffs*, then $\Gamma \Vdash (\alpha \supset \beta)$ if, and only if, $\Gamma \nVdash \alpha$ or $\Gamma \Vdash \beta$.[1]
- $\Gamma \Vdash \Box\, \alpha$ only if for every $\Delta \in W$, $\Gamma R \Delta$ implies that $\Delta \Vdash \alpha$.
- $\Gamma \Vdash \Diamond\, \alpha$ only if there exists a $\Delta \in W$ such that $\Gamma R \Delta$ and $\Delta \Vdash P$.

Together $\langle W, R, \Vdash \rangle$ represents a **propositional modal model** which McCance generally shortens to 'model'. (p. 4)

Note that first five cases above are identical to the truth table semantics of propositional calculus, with only the last two adding something new. We have already seen in Defn. 1 that we can rewrite $\Diamond$ in terms of the negation of $\Box$ and *vice versa*. *E.g.* if $\Diamond P$ were false in a world, this would mean that P is false in every accessible world, so the negation of P would be true in all possible worlds, *i.e.* $\Box\sim P$ would be true. Hence, $\Diamond P \equiv \sim\Box\sim P$. It is useful to think of the analogous case of why $(\exists x)Px$ is equivalent to $\sim(\forall x)\sim Px$. *I.e.* to say that there is an x such that Px is the same as to say that it is false that for all x, Px is false. Similarly, to say that $\Diamond X$ is the same as to say that it is false that X is false in every accessible world. (p. 4 - 5)

Having provided a clear interpretation of our symbols, McCance defines what it means for a formula to be tautologous or to use his term 'valid' in a given system.

Defn. 2.4 (**L**-valid). Let $\langle W, R, \Vdash \rangle$ be a model. We say that is model is 'based on the frame $\langle W, R \rangle$'. Let α be a *wff* then α is 'valid in $\langle W, R, \Vdash \rangle$' if $\Gamma \Vdash \alpha$ for every $\Gamma \in$ W. α is 'valid in the frame $\langle W, R \rangle$' if it is valid in every model based on $\langle W, R \rangle$. Finally, if α is valid in a collection of frames **L**, then we say it is **L**-valid. (p. 5)

[1] Please satisfy yourself by means of a truth table that $\alpha \supset \beta$ is logically equivalent to $\sim\alpha \vee \beta$.

Given a frame is just a set with a relation it is possible to choose frames based on the properties of R. (See Critical reasoning 14 on the logic of relations.) The first system **K**, named in honour of Saul Kripke, places no restrictions on the frame. McCance lists several other common systems together with their frame conditions, reproduced below.

Logic	Frame Conditions
K	no conditions
D	serial (For every world $\Gamma \in W$ there exits at least one $\Delta \in W$ such that Δ is accessible to Γ.)
T	reflexive
B	reflexive, symmetric
K4	transitive
S4	reflexive, transitive
S5	reflexive, symmetric, transitive

Table of Common Systems of Modal Logic and the Restrictions they Place on the Relations between Frames (*after McCance, 2008 p. 5*)

Note that these systems are interconnected. For example, any formula that is **K**-valid will be valid in all of these systems because its truth does not rely on any particular frame structure. Similarly, what is true in **B** will be true in **S5** since they share the same structure with the exception of **B** not necessarily being transitive. (p. 5)

We now have a systematic method of writing out statements and deciding which of them are true and under what circumstances; however we do not want to draw up truth tables for every possible world in order to decide on the veracity of every statement. Instead it would be much more desirable to formalise the logical process so that we begin with what we already know and apply rules and axioms to generate new theorems. McCance begins with system **K** because it makes no assumptions about frame structure and because we can always augment it later with further axioms to transform it into any of the other systems. He begins with a definition of a proof – a definition with which we are already familiar. (p. 5 - 6)

Defn. 2.5 An axiomatic 'proof' is a finite sequence of formulae, each of which is either an axiom or else follows from the earlier terms of the sequence by one of the rules of inference. An axiomatic 'theorem' is the last line of a proof. (p. 6)

The axioms are as follows:

Defn. 2.6 There are two classes of axioms for **K**

Classical Tautologies: All valid formulae of the Propositional Calculus (*i.e.* all the tautologies of traditional propositional logic) will be taken as axioms.

Schema **K**: For any *wffs* α and β, we assume that

$$\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$$

We are already familiar with rules of inference however; McCance provides one particular formal definition.

Defn. 2.7 A 'rule of inference' is an ordered pair (Γ, α), where Γ is a set of *wffs* and α is a single *wff.* If the propositions of Γ are theorems of the system, then so is α.

K has two rules of inference that can be written according to the above definition.

- *Modus ponens*: $(\{\alpha, \alpha \supset \beta\}, \beta)$ and
- Necessitation: $(\{\alpha\}, \Box\alpha)$. (p. 6)

McCance is at pains to ensure that the axioms we are introducing are reasonable. We ought to require no convincing that the tautologies and *modus ponens* of the Propositional Calculus are reasonable; however we might need some persuasion concerning Schema **K** and necessitation. Starting with Schema **K**: $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$; suppose we have a proof for $\Box(X \supset Y)$ where we interpret $\Box$ in terms of necessity. That means that we have proved that $X \supset Y$ is necessarily true. If we can show that X is necessarily true then we shall have shown that X is true in every possible world. But since $X \supset Y$ must be true in every possible world, it is reasonable to say that Y must be also, and hence that Y is necessarily true. Similarly, on the epistemic interpretation of $\Box$, we can say that if we know that $X \supset Y$ and we know that X, then we also know that Y. (p. 6)

Note that since we are in a proof system, X does not mean what it ordinarily does that 'X is the case.' Rather we mean that 'X is provable.' In the case of the logic of necessity, if we claim to have a proof of X that follows from other axioms and rules of inference, then we are claiming that X must necessarily be true. Similarly, in the case of epistemic logic, if we claim to have a proof for X, we can say that we know X to be both true and justified. "So Necessitation seems a reasonable axiom to adopt." (p. 6 - 7)

Before providing an example of an axiomatic proof, McCance introduces the Derived Rule of Regularity. We have already used derived rules extensively in Critical Reasoning 21 where they were treated as rules of inference once they had been deduced from axioms and rules of inference already available.

Defn. 2.8 Derived Rule of Regularity: $\{X \supset Y\}, \Box X \supset \Box Y$

Proof:

1. $X \supset Y$		
2. $\Box(X \supset Y)$	1. Necessitation	
3. $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$	Axiom **K**	
4. $\Box X \supset \Box Y$	3,2 M.P.	(p. 7)

The Derived Rule of Regularity can now be used in an example of an axiomatic proof.

***E.g.* 2.9** Prove $\Box(X \bullet Y) \supset (\Box X \bullet \Box Y)$

Proof:

1. $(X \bullet Y) \supset X$	Tautology (Axiom 2 of R.S.)
2. $\Box(X \bullet Y) \supset \Box X$	1 Regularity
3. $(X \bullet Y) \supset Y$	Tautology
4. $\Box(X \bullet Y) \supset \Box Y$	3. Regularity

5. $[\Box(X \bullet Y) \supset \Box X] \supset$
$\{[\Box(X \bullet Y) \supset \Box Y] \supset [\Box(X \bullet Y) \supset (\Box X \bullet \Box Y)]\}$ Tautology
6. $[\Box(X \bullet Y) \supset \Box Y] \supset [\Box(X \bullet Y) \supset (\Box X \bullet \Box Y)]$ 5, 2 M.P.
7. $\Box(X \bullet Y) \supset (\Box X \bullet \Box Y)$ 6, 4 M.P.

As mentioned before, the other systems in the table above are identical to **K** with just a few added restrictions on the frames. According to McCance, "Axiomatically, these systems simply add more axiom schemes." These added axioms are listed below left, with the common systems of modal logic formed by adding combinations of these schemes to **K**, shown below right.

Name	Scheme	Logic	Added axioms
D	$\Box P \supset \Diamond P$	**D**	D
T	$\Box P \supset P$	**T**	T
4	$\Box P \supset \Box\Box P$	**K4**	4
B	$P \supset \Box \Diamond P$	**B**	T, B
		S4	$T, 4$
		S5	$T, 4, B$

Joint Table of Common Axiom Schemes for Modal Proof Systems, Left and Common Logics and the Added Axioms Schemes Required to Axiomatize Them, Right. (after McCance, 2008 p.7- 8)

Soundness and Completeness

As with previous systems of proof, we want to know of those introduced by McCance so far:

a) whether the theorems so proved are valid in our collection of frames (*i.e.* soundness), and
b) whether the system is strong enough to derive all valid statements in our collection of frames (*i.e.* completeness).

Since we can already write all other operators in terms $\Box$, $\sim$, $\supset$ and $\bullet$; in the interests of economy we can restrict our demonstration to just these operators in the knowledge that others can be fleshed out later. (p. 8)

3.1 Soundness: According to McCance, "If our axioms are valid and our rules of inference never derive an invalid statement from a valid one, then clearly every provable formula will be valid."

Theorem 3.1 (**K** is sound). If X has a proof using the axiom system **K**, then X is **K**-valid.

Proof:[2] First consider the rules of inference.

Modus Ponens: Assume that the *wffs* X and $X \supset Y$ are **K**-valid. Therefore for all models based on all frames $\langle W, R\rangle$ and for all $\Gamma \in W$, $\Gamma \Vdash X$ and $\Gamma \Vdash X \supset Y$. Fix[3] a frame $\langle W, R\rangle$ and

[2] Note that where we endorse McCance's proofs we have reproduced them verbatim with only minor editing.
[3] By 'Fix' McCance simply means, 'consider a particular thing that won't change within the current context.'

a world $\Gamma \in$ W. Since we know that $\Gamma \Vdash X \supset Y$, we know that either $\Gamma \nVdash X$ or $\Gamma \Vdash Y$. But since we know that $\Gamma \Vdash X$ it follows that $\Gamma \Vdash Y$. Since our choice of world, model and frame within **K** were all arbitrary, Y is **K**-valid. Therefore *modus ponens* is a valid rule of inference. (p. 8)

Necessitation: Assume that X is **K**-valid. Therefore for all models based on frames $\langle W, R\rangle$ and for all worlds $\Gamma \in$ W, $\Gamma \Vdash X$. Fix again a frame $\langle W, R\rangle$ and a world $\Gamma \in$ W. Next consider a $\Delta \in W$ such that $\Gamma R \Delta$. But since X is true in every world, $\Delta \Vdash X$ and X is true in every world accessible to Γ. Therefore $\Gamma \Vdash \Box X$. Since our choices of frame and world were both arbitrary, $\Box X$ is **K**-valid. Therefore necessitation is also a valid rule of inference. (p. 8)

Finally, according to McCance, "All that remains is to prove our axioms valid." Recall that what he means by 'valid' here, we mean by 'tautological'. This much is trivial for tautologies involving non-modal statements because they are identical to those of the Propositional Calculus, for which we already have such techniques. Therefore the only remaining task is to show the validity of Schema **K**. (p. 8)

Schema **K**. We must show that $\Box(X \supset Y) \supset (\Box X \supset \Box Y)$ is true for all worlds in all frames. Let $\langle W, R\rangle$ be a frame and Γ a world in W. Assume that $\Box(X \supset Y)$ is true in Γ. Therefore, if $\Gamma R \Delta$ then $\Delta \Vdash(X \supset Y)$. If we want to show that $\Box X \supset \Box Y$ will be true in Γ then it must always be the case that either $\Box X$ is not true in Γ or $\Box Y$ *is* true in Γ. Assuming that $\Box X$ is true in Γ, then for all $\Delta \in W$, $\Gamma R \Delta$ implies that $\Delta \Vdash X$. If we fix such a Δ, then since $\Box(X \supset Y)$ is true in Γ and $\Gamma R \Delta$ we know that $\Delta \Vdash (X \supset Y)$. Hence Y must be true in Δ and therefore $\Box Y$ must be true in Γ. "So in every world Γ in any frame, either $\Box(X \supset Y)$ is false or $\Box X \supset \Box Y$ is true. Hence the axiom is **K**-valid." Since our axioms are all sound tautologous and our rules of inference reliably generate sound theorems from other sound theorems, every provable X in the axiom system **K** is **K**-valid. (p. 8 - 9)

According to McCance, proofs of the soundness of the other systems are largely corollaries of the one above. Since the other systems already share the same rules of inference and only add one or more axioms, all that needs to be done is to show that the axiom schemes are **L**-valid, where **L** is the collection of frames in question. (p. 9)

Theorem 3.2 (**D**, **T**, **K4**, **B**, **S4** and **S5** are sound.) Let **L** be **D**, **T**, **K4**, **B**, **S4** or **S5**. If X has a proof in the axiom system **L**, then X is **L**-valid.

Proof: We must show that each added axiom scheme is **L**-valid in the appropriately structured frame.

For D: $\Box P \supset \Diamond P$ which is the same as $\Box P \supset \sim\Box\sim P$. We must show that $\Box P \supset \sim\Box\sim P$ is true in all serial frames, *i.e.* where every world is related to at least one other world. Let $\langle W, R, \Vdash\rangle$ be a model and let Γ be a world in W. If $\Box P$ is true inΓ, then if $\Gamma R \Delta$, P will be true in Δ. But we want to show that $\sim\Box\sim P$ is true in Γ. "In other words, that there exists a Δ accessible to Γ such that $\Delta \Vdash P$."

Since $\langle W, R\rangle$ for **D** is serial, there exists at least one world Δ accessible to Γ. Fix such a Δ. Since we supposed that $\Box P$ is true in Γ and $\Gamma R \Delta$, we know that $\Delta \Vdash P$. Hence $\sim P$ must be false in Δ, so there exists a world accessible to Δ in which $\sim P$ is false. Therefore $\Box\sim P$ is not true in Γ, which is the same to say that $\sim\Box\sim P$ is true in Γ. Since our frame and choice of world were arbitrary, $\Box P \supset \sim\Box\sim P$ and hence $\Box P \supset \Diamond P$ is **D**-valid. And since D was the only added axiom of **D**, **D** must be sound.

For T: $\Box P \supset P$. Let $\langle W, R\rangle$ be a reflexive frame, *i.e.* for all $\Gamma \in W$, Γ is accessible to itself. Fix such a Γ and assume that $\Box P$ is true in Γ. Therefore P must be true in every world accessible to Γ and since Γ is accessible to itself, P must be true in Γ. Hence T is true in all reflexive models, including **T**. And since T was the only added axiom of **T**, **T** must be sound.

For 4: $\Box P \supset \Box\Box P$. Let $\langle W, R\rangle$ be a reflexive frame and let Γ be a world in W. Assume that $\Gamma \Vdash P$. Thus for every Δ accessible to Γ, P will be true in Δ. Either Δ is related to another world Θ or it is not. If it is not so related then $\Box P$ is vacuously[4] true in Δ. Say, on the other hand that $\Delta R\Theta$, then since our model is transitive, $\Gamma R\Theta$. Now since $\Box P$ is true in Γ, P must be true in Θ. But since our choice of Θ was arbitrary, P is true in all worlds related to Δ. Therefore $\Box P$ is true in every Δ related to Γ, hence $\Box\Box P$ is true in Γ. Therefore 4 is true in all transitive frames, including **K4**. But since 4 the only axiom added to **K4**, **K4** must be sound.

For **S4**, which is both reflexive and transitive, T and 4 are both valid in **S4**. And since T and 4 were the only axioms added in **S4**, **S4** is sound.

For B: $P \supset \Box \Diamond P$ which is the same as $P \supset \Box\sim\Box\sim P$. Let $\langle W, R\rangle$ be a symmetric, reflexive frame and let $\Gamma \in W$. Suppose that P is true in Γ. Since our frame is reflexive, there is obviously, at least, one world accessible to Γ. Let Δ be such a world, then since our frame is symmetric, Γ is accessible to Δ. We supposed that P is true in Γ, so $\sim\Box\sim P$ is true in Δ. This is true for all worlds accessible to Γ, therefore $\Box\sim\Box\sim P$ is true in Γ. B is therefore valid in all frames that are both symmetric and reflexive, including **B**. Since **B** is reflexive, T is also valid in **B**. And since T and B were to only added axioms, **B** is sound.

Finally, for **S5** the frames must be reflexive, symmetric and transitive, therefore T, 4 and B are all valid in **S5**. And since these were the only added axioms, **S5** is sound. (p. 9 - 10)

3.2 Completeness. According to McCance, the proof of completeness is significantly more complex than that for soundness. Beginning with some preliminary definitions and results that work for any system **L**, he sets out a proof for the completeness of **K**, leaving the remaining systems as corollaries, as before. (p. 10)

Defn. 3.3 Consistency. Let $\perp$ be an abbreviation for the contradiction ($P \bullet \sim P$). A finite set of formulae $\{X_1, X_2, \dots, X_n\}$ is **L**-consistent if $(X_1 \bullet X_2 \bullet \dots \bullet X_n) \supset \perp$ is *not* provable using the **L** axiom system. An infinite set is **L**-consistent if every finite subset is **L**-consistent.

Defn. 3.4 Maximal consistency. A set of formulae S is maximally **L**-consistent if S is **L**-consistent and no proper extension of it is **L**-consistent. *I.e.* if $S \subseteq S'$ (read S is a subset of or equal to S') and S' is also **L**-consistent, then $S = S'$. (p. 10)

---

[4] A vacuous truth is a statement that asserts that all members of the empty set have a certain property. *E.g.* if someone asserts that there are no cell phones switched on in the room, his or her statement will be vacuously true if there were no cell phones in the room anyway.

Theorem 3.5 If S is **L**-consistent, then it can be extended to a maximally **L**-consistent set.[5]

Proof: The general strategy for this proof is to add either formulae or their negations one at a time to a consistent set until there are no more formulae to be added. The resultant set will be consistent because every finite subset of it was consistent. It will also be maximally consistent since every formula or its negation will already be in the set.

Let S be an **L**-consistent set of formulae. Since every formula is a finite combination of a finite set of symbols, we know from set theory that the set of all *wffs* is countable. We can therefore map them onto the natural numbers like so: $X_1; X_2; X_3 \ldots$ We can use this fact to construct a sequence of sets $S_0; S_1; S_2; S_3 \ldots$ as follows:

$$S_0 \quad = S$$

$$S_{n+1} = \begin{cases} S_n \ \cup \ X_{n+1} \text{ if } S_n \ \cup \ X_{n+1} \text{ is consistent} \\ S_n \text{ otherwise.} \end{cases}$$

Note first of all that by construction, each S_i is consistent. Furthermore, $S_i \subseteq S_{i+1}$ for every $i \in \mathbb{N}$. We can therefore define the limit of this sequence S^* as $S_0 \cup S_1 \cup S_2 \ldots$ Our claim is that S^* is maximally consistent - *i.e.* S^* is **L**-consistent and if $S^* \cup S'$ is consistent then $S^* = S'$.

S^* is **L**-consistent. We can show this by *reductio ad absurdum*. Assume that S^* is not **L**-consistent. Then the exists some finite subset $Z = \{Z_1;\ Z_2;\ Z_3 \ldots\ Z_n\}$ of S^* such that $(Z_1 \bullet Z_2 \bullet \ldots \bullet Z_n) \supset \bot$. But we know that not every Z_i came from S since S was **L**-consistent. We also know that for every $Z_i \notin S$ there exists some $j \in \mathbb{N}$ such that $Z_i \notin S_j \cap S_{j-1}$. (*I.e.* S_j is the first set in the sequence containing Z_i) Since the set of Zs is finite we can let k equal to the greatest such j. As indicated above $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k$. Since each Z_i is in S_j for some $j \leq k$, it follows that $Z_i \in S_k$. Therefore S_k is inconsistent with **L**. However we know that S_i is **L**-consistent for every $i \in \mathbb{N}$, therefore our assumption must have been false, therefore S^* *is* **L**-consistent. (p. 10 - 11)

We can prove that S^* is maximally consistent by *reductio ad absurdum*. Assume that there was a proper extension of S^* that maintained **L**-consistency. Then there is some formula $Z \notin S^*$ such that $S^* \cup \{Z\}$ is **L**-consistent. But we know that our list of formulae $X_1; X_2; X_3 \ldots$ contains all formulae, so there exists an $i \in \mathbb{N}$ such that $Z = X_i$. So either $X_i \in S_i$ or $X_i \notin S_i$. If $X_i \in S_i$ then $X_i \in S^*$ and $S^* \cup \{X_i\}$ is not a proper extension of S^*. But if $X_i \notin S_i$ then it must be that $S_{i-1} \cup X_i$ was not consistent. Since $S_{i-1} \subseteq S^*$ that would make $S^* \cup \{X_i\}$ not consistent. So our assumption must have been false and there cannot be a proper extension of S^* that maintains **L**-consistency. Therefore S^* is maximally consistent. (p. 11)

Theorem 3.6 If the set $\{\sim\Box B;\ \Box A_1;\ \Box A_2\ \ldots\}$ is **L**-consistent, then so is $\{\sim B;\ A_1;\ A_2; \ldots\}$.

Proof: We proceed by contraposition.[6] Assume that $\Gamma = \{\sim B;\ A_1;\ A_2; \ldots\}$ is inconsistent with **L**. This means that there is some finite $\Delta \subseteq \Gamma$ whose elements imply a contradiction. Since any

[5] McCance's proof of this and subsequent theorems have been adapted from Fitting & Mendelsohn (1998).

extension of an inconsistent set is itself inconsistent, we can assume that Δ contains $\sim B$ and reorder any A_is such that $\Delta = \{\sim B;\ A_1;\ A_2; \ldots;\ A_n\}$. From this we generate the following proof.

1. $(\sim B \bullet\ A_1 \bullet A_2 \bullet \ldots \bullet A_n) \supset \bot$	Assumption
2. $(A_1 \bullet A_2 \bullet \ldots \bullet A_n) \supset (\sim B \supset \bot)$	1 Exportation
3. $(A_1 \bullet A_2 \bullet \ldots \bullet A_n) \supset B$	$(\sim X \supset \bot)\ \equiv X$
4. $\Box((A_1 \bullet A_2 \bullet \ldots \bullet A_n) \supset B)$	3 Necessitation
5. $\Box(A_1 \bullet A_2 \bullet \ldots \bullet A_n) \supset \Box B$	4 Schema **K**
6. $(\Box A_1 \bullet \Box A_2 \bullet \ldots \bullet \Box A_n) \supset \Box B$	5 Theorem 2.9
7. $(\Box A_1 \bullet \Box A_2 \bullet \ldots \bullet \Box A_n) \supset (\sim\Box B \supset \bot)$	$(\sim X \supset \bot) \equiv X$
8. $(\sim\Box B \bullet \Box A_1 \bullet \Box A_2 \bullet \ldots \bullet \Box A_n) \supset \bot$	7. Exportation

So the set $\{\sim\Box B \bullet \Box A_1 \bullet \Box A_2 \bullet \ldots\ \}$ has an inconsistent finite subset and is therefore inconsistent itself. Thus the theorem is proven by contraposition. (p. 11)

Defn 3.7 Canonical Model. We define the 'canonical model of **L**' $\langle W, R, \Vdash\rangle$ as follows. Let W be the set of all maximally **L**-consistent sets of *wffs*. We define the binary relation R on W as follows. Let Γ, $\Delta \in W$. We say that $\Gamma R \Delta$ if and only if for every proposition of the form $\Box X$ in Γ, X is true in Δ. Finally if P is a propositional variable, $\Gamma \Vdash P$ if and only if $P \in \Gamma$. (p. 11)

Theorem 3.8 Truth Lemma. Given a canonical model of **L** $\langle W, R, \Vdash\rangle$, a world $\Gamma \in W$, and a *wff* α, the following is true:

$$\Gamma \Vdash \alpha \text{ if and only if } \alpha \in \Gamma.$$

Proof: Let Γ be a world in the canonical model of **L** and let α be a *wff*. We proceed by induction on the complexity of α.

Negations. Say that our formula is $\sim\gamma$, where γ is true in Γ if and only if $\gamma \in \Gamma$. Suppose that $\Gamma \Vdash \sim\gamma$, then $\Gamma \nVdash \gamma$, so $\gamma \notin \Gamma$. Since Γ is maximally consistent, it follows that $\sim\gamma \in \Gamma$. Now suppose that it is the case that $\sim\gamma \in \Gamma$, then since Γ is consistent $\gamma \notin \Gamma$. By our assumption it follows that γ is not true in Γ, therefore $\Gamma \Vdash \sim\gamma$. (p. 11)

Modalities. Say that our formula is $\Box\gamma$. First assume that $\Box\gamma \in \Gamma$ and that Δ is accessible to Γ. By our definition of R above we know that $\Delta \Vdash \gamma$. Since our choice of Δ was arbitrary, this is true for all worlds to which Γ is related. Therefore $\Box\gamma$ is true in Γ.
Next assume that $\Box\gamma \notin \Gamma$. Since Γ is maximally consistent, this means that $\sim\Box\gamma \in \Gamma$. Now consider all statements beginning with $\Box$, *i.e.* $\{\Box X_1; \Box X_2; \ldots\}$. We know from Theorem 3.6 above that if $\{\sim\Box\gamma; \Box X_1; \Box X_2; \ldots\}$ is consistent then so is $\{\sim\gamma;\ X_1;\ X_2 \ldots\}$. We can extend

---

[6] Contraposition is a form of inference that says a conditional statement $P \supset Q$ is logically equivalent to its contrapositive $\sim Q \supset \sim P$. *E.g.* 'If it is a cat then it is a mammal' is logically equivalent to 'If it is not a mammal then it is not a cat.' (Wikipedia: Contraposition)

this to form a maximally consistent set Δ. Now for every statement of the form $\Box P \in \Gamma$, P must be in Δ. So by our inductive hypothesis, $\Delta \Vdash P$. Since for every $\Box P \in \Gamma$, $\Delta \Vdash P$, we know that $\Gamma R \Delta$ by definition 3.7. But since $\sim\gamma$ is true in Δ it must be that $\Box\gamma$ is false in Γ. So by contraposition if $\Box\gamma$ is true in Γ, then $\Box\gamma \in \Gamma$. (p. 11 - 12)

Implications. Say that our formula is $\gamma \supset \delta$ and that γ and δ are each true if and only if they are elements of Γ. Assume that $\Gamma \Vdash \gamma \supset \delta$. Then either γ is false in Γ or δ is true. In the case that γ is true, δ must also be true, therefore both γ and δ are in Γ. And since both γ and δ are elements the maximally consistent set, $\gamma \supset \delta$ must be in that set too.
Assuming that $\gamma \supset \delta \in \Gamma$, then either $\sim\gamma$ or δ is in Γ as well, since Γ is maximally consistent. So either $\Gamma \Vdash \sim\gamma$ or $\Gamma \Vdash \delta$ therefore $\gamma \supset \delta$ is true in Γ. (p. 12)

Conjunctions. Say that our formula is $\gamma \bullet \delta$ and that γ and δ are each true if and only if they are elements of Γ. Assume that $\Gamma \Vdash \gamma \bullet \delta$, then both γ and δ must be true, therefore $\gamma, \delta \in \Gamma$.
Assuming that $\gamma, \delta \in \Gamma$, then $\Gamma \Vdash \gamma$ and $\Gamma \Vdash \delta$, therefore we know by definition that $\Gamma \Vdash \gamma \bullet \delta$. (p. 12)

With these definitions, theorems and proofs in place, we can finally show that **K** is complete.

Theorem 3.9 If we assume that X is **K**-valid, then there is a proof of X in the axiom system **K.**

Proof: Proceeding by contraposition, assume that X has no proof in **K**. In that case $\{\sim X\}$ will be consistent. And since we cannot prove X, neither could we prove $X \bullet \sim X$.

Extend X to a maximally consistent set X^* letting $\langle W, R, \Vdash\rangle$ be the canonical model of **K**. Since X^* is maximally consistent it must be in W. Since $\sim X \in X^*$, $\sim X$ must be true in X^*. And since X^* is maximally consistent, $X \notin X^*$. Therefore X is not true in X^*, meaning that X is not valid in the canonical model.

Clearly the canonical model of **K** is in the collection of frames **K**, since **K** places no restrictions on its fames. Therefore we have shown that there exists a model based on a frame in **K** in which X is not valid and hence not **K**-valid.

To show that the remaining systems are complete merely requires that the canonical model of a given system **L** meets the requirements of that system. Thus, for example, the proof of completeness for **T** consists in showing that the canonical model of **T** is based on a reflexive frame. Before proceeding with the remaining completeness poofs McCance introduces a useful lemma.

Lemma 3.10 Let $\langle W, R\rangle$ be the canonical model for a system **L** and let $\Gamma \in W$. If Γ contains no statements beginning with $\Box$, then $\Gamma R \Delta$ for all $\Delta \in W$.

Proof: Fix $\Delta \in W$ and let Σ be the set of all statements beginning with $\Box$ in Γ. Now define $\Phi = \{X | \Box X \in \Sigma\}$. From the definition of the canonical model we know that $\Gamma R \Delta$ only if $\Phi \subseteq \Delta$. Since, by hypothesis Γ contains no statements beginning with $\Box$, Σ must be empty and so must Φ. Therefore $\Phi = \emptyset \subseteq \Delta$. Hence Δ is accessible to Γ.

Theorem 3.11 The systems **D**, **T**, **K4**, **B**, **S4** and **S5** are all complete.

Proof: The strategy here is to show that for any of the above axiom systems **L**, the frame on which the canonical model is based is in the collection of frames **L**. In each of the following proofs let $\langle W, R, \Vdash \rangle$ be the canonical model of **L** under consideration.

D. Fix $\Gamma \in W$. Since D ($\Box P \supset \Diamond P$ which is the same as $\Box P \supset \sim\Box\sim P$) is an axiom of **D**, we know that $(\Box P \supset \sim\Box\sim P) \in \Gamma$ and that $\Box P \supset \sim\Box\sim P$ is true in Γ according to the Truth Lemma (Theorem 3.8). Now assume that there is no $\Delta \in W$accesable to Γ and let X be a propositional variable. Since there are no worlds accessible to Γ, $\Box X$ is vacuously true. However since D is true in Γ we know that $\sim\Box\sim X$ is true in Γ, since Γ is maximally consistent. Therefore $\Box\sim X$ must be false in Γ. But this can only be the case if there exists a Δ accessible to Γ such that $\Delta \nVdash \sim X$. This however contradicts our assumption which must therefore be wrong. So there must exist a Δ accessible to Γ. Since this is true for all $\Gamma \in W$, the canonical frame of **D** is serial.

T. Fix $\Gamma \in W$. Since T ($\Box P \supset P$) is an axiom of **T** we know that T is in Γ and hence that $\Box P \supset P$ is true in Γ. If there are no statements in Γ beginning with $\Box$, then $\Gamma R \Gamma$ is automatically true. But if there are such statements in Γ then we can let $\Sigma = \{\Box X_1;\ \Box X_2;\ \dots\}$ be the set of all statements in Γ beginning with $\Box$. Since T is true in Γ each X_i is true in Γ as well. So by the definition of canonical models, $\Gamma R \Gamma$ which is reflexive. And since our choice of Γ was arbitrary, the frames of **T** are reflexive.

K4. Let Γ, Δ, $\Theta \in W$. Since 4 ($\Box P \supset \Box\Box P$) is an axiom of **K4** w know that it must be an element of Γ and by the truth lemma that $\Gamma \Vdash \Box P \supset \Box\Box P$. Assume that $\Gamma R \Delta$ and $\Gamma R \Theta$ are true. If there are no statements in Γ beginning with $\Box$, then $\Gamma R \Theta$ must be true. But if there are such statements in Γ then we can let $\Sigma = \{\Box X_1;\ \Box X_2;\ \dots\}$ be the set of all statements in Γ beginning with $\Box$. Since 4 is true, we know that $\Box\Box X_i$ is true in Γ for each $i \in \mathbb{N}$. Now fix such an i. As Δ is accessible to Γ, it follows that $\Box X_i \in \Delta$. Since $\Box X_i$ must be true in Δ and in $\Gamma R \Theta$, it must be the case that $X_i \in \Theta$. So by the definition of canonical models, Θ must be accessible to Γ. Therefore the canonical model of **K4** is transitive.

B. T and B ($P \supset \Box \Diamond P$ which is the same as $\Box\sim\Box\sim P$) are both axioms of **B**. And since we already know that T implies that $\langle W, R \rangle$ is reflexive, all that remains is to show that **B** guarantees symmetry. We can show this by contraposition. Let $\Gamma, \Delta \in W$ and assume that $\Delta R \Gamma$. Now suppose that $X \notin \Delta$. Since Δ is maximally consistent it follows that $\sim X$ is in Δ. But since B is an axiom, we know that $\Box\sim\Box\sim P$ is true in Δ. And since Γ is accessible to Δ, this means that $\sim\Box X$ is true in Γ. However we know that $\Box X$ is not in Δ, so by contraposition if $\Box X$ is in Γ then X is in Δ. Therefore Δ is accessible to Γ by definition, making the canonical model of **B** symmetric.

S4. Since the axiom system **S4** includes T and 4, which we have already shown to guarantee reflexivity and transitivity respectively, we already know that the canonical model of **S4** is both reflexive and transitive.

S5. Since the axiom system **S5** includes T, B and 4, which we have already shown to guarantee reflexivity, symmetry and transitivity respectively, we already know that the canonical model of **S5** is reflexive, symmetrical and transitive. (p. 13)

Quantified Modal Logic

In Critical Reasoning 11 we introduced predicate logic as an extension of sentential logic as a means by which to represent quantification using the universal (∀) and existential (∃) operators. However adding such quantifiers to modal logic is more complex and sometimes attended with philosophical problems. The language for quantified modal logic is as follows:

- The same operators and syntax as propositional modal logic;
- The quantifiers ∀ and ∃;
- As many variables as needed, represented by lowercase Roman letters;
- As many n-place predicates or relational symbols as needed, represented by capital Roman letters;
- A special relation =, the identity relation.

Atomic formulae are represented by $R_{x_1;\, x_2 \ldots x_n}$ where R is an n-ary predicate or relational symbol. Compound formulae are constructed form atomic formulae and logical connectives in the usual way. Since we are interested in adding quantification, it is useful to have a model consisting of some non-empty set over which to quantify. Call this set the 'domain'. To determine the truth of a formula like $(\forall x)(Px)$ in a world Γ requires that we check to see that Px is true in Γ for every x in the domain. Similarly, to determine the truth of modalities such as $\Box(\forall x)(Px)$ requires that we check to see that Px is true for every x in every accessible world. Likewise, to determine the truth of modalities such as $(\exists x)(Px)$ requires that that there be some x in the domain for which Px is true. (p. 14)

There is some difficulty in talking about modal identity statements such as $(\exists x)[\Box(x = x)]$ in a world Γ. Ostensively, this says that there exists an $x \in \Gamma$ such that in every accessible world Δ, x is identical to x. That much is seemingly obvious; how could it be that something is not identical to itself? But what if x does not exist in Δ? Can our logic handle statements about objects that don't exist? On the other hand, we don't want to assert $x \neq x$, for that is a contradiction. (p. 14)

One potential solution is to simply require that if x exists in one world's domain, then it must exit in all of them. This is known as a **constant domain model**, the frame of which is a collection of worlds W, an accessibility relation R, and a single non-empty set D. However there are cases which require that domains vary from world to world. In temporal logic, for example, it would be unreasonable to require that every object that exists now must also have existed in the distant past. Thus temporal logic is better suited to a **varying domain model**, the frame of which is identical to that of a constant domain model: $\langle W, R, D \rangle$. In this case however D is a function that assigns each world in W to a domain so that the actual domain for a world Γ is not D but $D(\Gamma)$. (p. 14)

According to McCance, this simply replicates the problem of deciding whether $x = x$ is true in Δ if $x \notin D(\Delta)$. One way out of the conundrum is to simply allow that some statements are neither true nor false, as is allowed in some multivalent logics. This results in what is called a **partial model**.

Another workaround is to define a **frame domain** F, where F is the union of all domains. This allows us to talk about a fixed x irrespective of whether it exists in the world of discourse. In this way we can define '$=$' so that x is still identical to x, whether or not it exists in a particular world. Thus the relation '$=$' can be defined as the set of all ordered pairs $(y; y)$ such that $y \in F$. This has the virtue of allowing us to decide on the truth or falsity of Px in Δ even when $x \notin D(\Delta)$. Without the frame domain Px would be vacuously true. With the frame domain we can meaningfully talk about x not being in relation to P, in which case Px is false in the ordinary sense. (p. 14 - 15)

Task

You will not be asked to prove any theorems of modal logic as part of a graduate degree in philosophy. You will however be expected to follow and understand a number of proofs that involve modalities, especially within the philosophy of language. Zalta (1995) and McCance (2008) have provided brief historical introductions for modal logic which you may wish to consult. Provide your own motivation for modal logic based on examples within your syllabus or study units posted here. You may wish to look ahead. There is no general feedback given since your task will be unique.

References

FITTING, M. & MENDELSOHN, R. (1998) *First-Order Modal Logic*. Kluwer Academic Publishers: The Netherlands

McCANCE, J. (2008) A Brief Introduction to Modal Logic. *Semantic Scholar* 2008 p. 1- 15

ZALTA, E. (1995) Basic Concepts in Modal Logic. Center for the Study of Language and Information Stanford University. http://mally.stanford.edu/notes.pdf