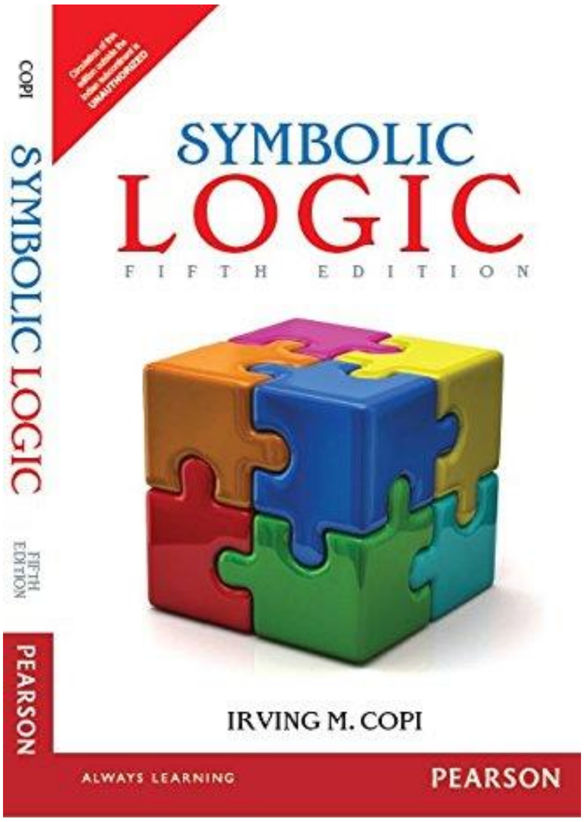
Critical Reasoning 21 - A Propositional Calculus

This study unit follows Copi's (1979) construction of a formal system or logical calculus "intended to be adequate for the formulation of arguments whose validity depends upon the ways in which sentences are truth functionally compounded." (p. 213) Up till now we have been working with symbolic logic as if the sentential or propositional calculus and then the predicate calculus were givens. Unlike natural languages, formal systems are expressed in a formal language that has to be constructed. Of course, we shall want to *talk about* this formal language, in which case it will be the *object* of our discussion and thereby designated as the object language. Initially our object language will consist of symbols and formulae that have no meaning or usage until we *give* them an interpretation using a different language. The language used to talk about the object language is known as the metalanguage, which in in this study unit will be English. Note that a natural language such as English or Japanese can be both an object language and a metalanguage according to our purpose. (p.213)



Copi's Symbolic Logic has been a Standard Textbook for at Least Three Generations of Philosophers. Now Available in Paperback

According to Copi, an object language may be discussed from various points of view. We may be interested in dialectical shifts in usage or pronunciation according to geographical location. Alternatively, we may be interested in the meaning or interpretation of a language such as when compiling a dictionary. For this purpose a lexicographer would require a *semantic* metalanguage. If, on the other hand, we were interested in describing the formal structure of a natural language as in a textbook of grammar or the development of theorems in an uninterpreted logistic system we would require a *syntactical* metalanguage also known as syntax language. (*l.c.*)

Copi emphasises that the terms 'object language' and 'metalanguage' are relative terms. Any language can be an object language when it is being talked about. Similarly, any language which must be interpreted or meaningful can be a metalanguage when it is used to discuss an object language. Any sufficiently rich or complex language can be used to express its own syntax and most of its semantics; however no consistent language can express all of the truth conditions of all of its statements. Consider the by now familiar syntactically correct and meaningful sentence:

This sentence is false. …①

As we have seen, this leads to a contradiction: ① is true if and only if ① is false and *vice versa*. But this is not simply sophistry; it reveals an important point. ① is being used both as a sentence of object language and a sentence in its own metalanguage. If the distinction is strictly adhered to, the contradiction evaporates. In the following sections Copi proceeds to construct his logistic system.

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Primitive Symbols and Well Formed Formulae

A logistic system requires two kinds of primitive symbols or basic units: 'propositional' symbols to stand for propositions and 'operator' symbols to stand for logical operators. Only four of the latter are required:

$$\bullet \quad \sim \quad (\quad)$$

We also require an infinite number of propositional symbols. Copi chose the first four capital letters of the Roman alphabet in bold, with or without subscripts, thus:

$$\begin{array}{lllll} \mathbf{A} & \mathbf{A_1} & \mathbf{A_2} & \mathbf{A_3} & \ldots \\ \mathbf{B} & \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} & \ldots \\ \mathbf{C} & \mathbf{C_1} & \mathbf{C_2} & \mathbf{C_3} & \ldots \\ \mathbf{D} & \mathbf{D_1} & \mathbf{D_2} & \mathbf{D_3} & \ldots \end{array}$$

Note that the points to the right represent continuation of the sequence, though they are not part of the system. Copi might have used only a single letter and a dash or a subscript to provide and infinite set of propositional symbols, such as **A**, **A'**, **A''**, **A'''**, **A''''**, … but these could be easily misread. In fact, he *could* have dispensed with '(' and ')' altogether despite their utility once we have given them an interpretation as parentheses. For now however, these symbols must be regarded as completely uninterpreted and without meaning. Yet they will be the only symbols that our system contains for proving theorems and deriving conclusions from premises *within* the system. Of course, Copi already had an idea of what interpretation he eventually wished to give these symbols within the system, which no doubt guided his choice of them in the first place. (p. 215)

Copi's intended interpretation also constrains his syntactical definition of a 'well formed formula' within the system. A formula within the system may be any sequence of symbols including,

$$\mathbf{B_1} \qquad \mathbf{(A)\bullet(A)} \qquad \mathbf{\sim(D)\sim(\sim)} \qquad \mathbf{\sim((A_1)\bullet(C_3))}$$

$$\mathbf{B_2B_3B_7\sim(\ )(\ )\bullet(\ )} \qquad \mathbf{)))((} \qquad \ldots$$

however not every formula counts as a well formed formula, the definition of which will have to be stated in our Syntax Language. Letters printed in lightface italic, without subscripts, 'A', 'B', 'C' and 'D' will distinguish those of our metalanguage from those of the object language. By convention, the meaning of the metalanguage symbols is the same as the meaning of the corresponding object language symbols. In addition it is necessary to introduce symbols that may denote any formula. These are known as 'propositional variables' and are represented by, 'P', 'Q', 'R', 'S', …, with or without subscripts. The symbols of propositional variables in the propositional calculus to be developed are uninterpreted and have no meaning; however the same symbols in our Syntax Language are interpreted and do have meaning. (p. 216)

Every propositional variable of Syntax Language denotes any formula in the object language, subject to the following restrictions:

> In any sentence or sequence of sentences of our Syntax Language, two distinct propositional variables, say 'P' and 'Q', may denote either two distinct formulas of our object language, for example '$\mathbf{B_1}$' and '$\sim$**((A$_1$)•(C$_3$))**', or one and the same formula of our object language. Although a propositional variable may denote *any* formula of the object language in any one context, it must continue to denote the same formula whenever it occurs in that context. Thus the propositional variables 'P', 'Q', 'R', 'S', …, of our metalanguage may have substituted for them *any name* in the metalanguage of *any formula* of the object language. (p. 216)

At this point Copi introduces the symbols '•', '$\sim$', '(' and ')' into the Syntax Language along with their meanings, subject to the above restrictions.

- Where any propositional variable, say 'P' of the Syntax Language denotes some formula in the object language, say '**A**', then the symbol '$\sim(P)$' of the Syntax Language will denote '$\sim$(**A**)' in the object language.

- Where any propositional variables, say 'P' and 'Q' of the Syntax Language denote two formulae in the object language, say '**A**' and '$\sim$($\mathbf{B_2}$)', respectively, then the symbol '(P)•(Q)' of the Syntax Language will denote '(**A**)•($\sim$($\mathbf{B_2}$))' in the object language. (*l.c.*)

It is not possible to give an exhaustive list of all well formed formulae (*wffs*) of the object language since there are infinitely many of them; instead it is possible to give a **recursive definition** (*i.e.* involving the repeated application of a rule, definition, or procedure) of them accordingly.

(a) Any propositional symbol is a *wff*.
(b) If any formula P is a *wff*, then $\sim(P)$ is a *wff*.
(c) If any formulae P and Q are both *wffs*, then (P)•(Q) is a *wff*.

(No formula of the object language is *wff* unless it can be derived from the above rules.[1])

Not only does this definition allow for the formation of infinitely many *wffs*, it also provides an effective criterion for recognising them, no matter how finitely long, in a finite number of steps. Consider Copi's example:

$\sim$**((A)•($\sim$B)))**

By (b) the formula is a *wff* if (A)•$(\sim B))$ is also a *wff*. By (c) the latter is a *wff* if A and $\sim B$ are also *wffs*. By (a) A is a *wff* and by (b) $\sim B$ is also a *wff*. Therefor the formula above is a *wff*. Whereas we earlier took the propositional variables, 'P', 'Q', 'R', 'S'… to represent any formula, we now restrict their use to only *wffs* of the object language. (p. 216 - 217)

Copi's exercises on p. 217 - 218 require that we use the recursive definition for *wffs* above to decide which formulae are *wffs* of the object language. Today this sort of task is usually handled by

[1] There are other recursive definitions to this one that would make different formulae *wffs*, however suitable logistic systems could equally well be developed from them. (See Copi's footnote 3 on p. 217)

computer, especially in programming languages; however it is worth gaining a little practice by attempting a selection of these exercises by hand. Here are the first three provided by Jon Ross:

1. $(\sim(\mathbf{A_1}))\bullet(\mathbf{A_1})$ is a *wff*, by (c), if $\sim(\mathbf{A_1})$ and A_1 are *wffs*

   $\sim(\mathbf{A_1})$ is a *wff*, by (b), if $\mathbf{A_1}$ is a *wff*

   $\mathbf{A_1}$ is a *wff*, by (a)

   $\therefore$ $(\sim(\mathbf{A_1}))\bullet(\mathbf{A_1})$ is a *wff*

2. $\sim(\sim((\mathbf{B_1})\bullet(\sim(\mathbf{C_3}))))$ is a *wff*, by (b), if $\sim((\mathbf{B_1})\bullet(\sim(\mathbf{C_3})))$ is a *wff*

   $\sim((\mathbf{B_1})\bullet(\sim(\mathbf{C_3})))$ is a *wff*, by (b), if $(\mathbf{B_1})\bullet(\sim(\mathbf{C_3}))$ is a *wff*

   $(\mathbf{B_1})\bullet(\sim(\mathbf{C_3}))$ is a *wff*, by (b), if $\mathbf{B_1}$ and $\sim(\mathbf{C_3})$ are *wffs*

   $\sim(\mathbf{C_3})$ is a *wff*, by (b), if $\mathbf{C_3}$ is a *wff*

   $\mathbf{C_3}$ is a *wff*, by (a)

   $\mathbf{B_1}$ is a *wff*, by (a)

   $\therefore$ $\sim(\sim((\mathbf{B_1})\bullet(\sim(\mathbf{C_3}))))$ is a *wff*

3. $\sim(\sim((\mathbf{B_2})\bullet(\sim(\mathbf{D_4})))$ is a *wff*, by (b), if $\sim((\mathbf{B_2})\bullet(\sim(\mathbf{D_4}))$ is a *wff*

   $\sim((\mathbf{B_2})\bullet(\sim(\mathbf{D_4}))$ is a *wff*, by (b), if $(\mathbf{B_2})\bullet(\sim(\mathbf{D_4})$ is a *wff*

   $(\mathbf{B_2})\bullet(\sim(\mathbf{D_4})$ is a *wff*, by (c), if $\mathbf{B_2}$ and $\sim(\mathbf{D_4}$ is a *wff*

   $\sim(\mathbf{D_4}$ is not a *wff*

   $\therefore$ $\sim(\sim((\mathbf{B_2})\bullet(\sim(\mathbf{D_4})))$ is not a *wff*

Ross has provided the remaining solutions to this exercise as well as those to the rest of the exercises in this chapter; however they can no longer be found in the original location.

When we come to provide an interpretation of our logistic system, we intend for its propositional symbols to stand for statements of English that contain only truth-functional components. Thus where P stands for any statement in English whatsoever, $\sim(P)$ stands for its negation. Similarly, where P and Q stand for any two English statements whatsoever, $(P)\bullet(Q)$ stands for their conjunction. Besides these we want our system to be able to express every truth-functional compound statement. In other words we want it to be **functionally complete** (as a special case of 'expressive completeness' mentioned in Critical Reasoning 16. (p. 218)

At this point Copi decides to give our system a name: 'R.S.' for 'Rosser's System' after the logician J. Barkley Rosser, then of Cornell University, from whose system it is derived. We have already shown that negation and conjunction can be expressed by R.S. but we are also familiar with disjunctions, conditionals and equivalences which must be considered. Where P and Q stand for any two English statements whatsoever, their *weak* disjunction (when either or both are true) can be expressed in R.S. by $\sim((\sim(P)) \bullet (Q)))$. However since this sort of disjunction is so common and its symbolisation, as such, a little unwieldly, we introduce into R.S. the abbreviation 'P v Q' into the Syntax Language to denote all *wffs* in the object language identical to those denoted by '$\sim((\sim(P)) \bullet (\sim(Q)))$.'[2] Strong disjunction (when either P or Q but not both are true) can be expressed in R.S. by $(\sim((P) \bullet (Q))) \bullet (\sim((\sim P)) \bullet (\sim(Q))))$. However since strong disjunction is not especially commonly used we need not introduce a special abbreviation for it. In Critical Reasoning 05 we encountered the 'exclusive or' sometime symbolised 'XOR'. While XOR logical gates are common in electrical circuits, the use of the exclusive or in logic is not so common but could just as easily have been introduced at this point. (p. 218)

Truth-functional conditional statements of the form 'if P… then Q' can be denoted by $\sim((P) \bullet (\sim(Q)))$. This too is so common as to justify the introduction the symbol '⊃' into the Syntax Language to denote all *wffs* in the object language identical to those denoted by '$\sim((P) \bullet (\sim(Q)))$'. Material equivalence between two statements that have the same truth values such that each implies the other can be symbolised $(P \supset Q) \bullet (Q \supset P)$. This denotes all *wffs* in the object language identical to those denoted by $(\sim((P) \bullet (\sim(Q)))) \bullet (\sim((Q) \bullet (\sim(P))))$. Since material equivalence is frequently used in philosophy and logic we can introduce the symbol '≡' into the Syntax Language by defining '$P \equiv Q$' as an abbreviation for '$(P \supset Q) \bullet (Q \supset P)$'. (p. 218 - 219)

In order to make the text of the metalanguage more easily readable Copi adopts a couple of notational conventions. The First is to dispense with the dot when it appears between adjacent closing and opening parentheses. Thus '$(P)(Q)$' denotes the same formula as '$(P) \bullet (Q)$' does in the object language. The second is to replace parentheses by square brackets or curly braces whenever they make reading easier. (p. 218)

Just as we learned in primary school arithmetic by the acronym BODMAS, assigning an order of precedence to operations can reduce reliance on too much punctuation and clear up ambiguities. By convention the following list of symbols of Syntax Language is ordered so that those on the left have greater order of precedence or greater scope than those to the right:

$$\equiv \quad \supset \quad \mathrm{v} \quad \bullet \quad \sim$$

Copi explains the gist of what is intended by way of examples. Without an agreed upon order of precedence '$\sim PQ$' would be ambiguous. Do we mean to negate P and then conjoin it to Q or do we mean to conjoin P and Q and then negate the conjunction? The connective '•' which is implied by the juxtaposition of P and Q has a greater order of precedence than that of '∼', therefore the scope of '•' extends over the symbol '∼'. Thus, we must negate P first and then conjoin it to Q. The expression 'P v QR' would also be ambiguous without an agreed upon order of precedence. However because 'v' has precedence over '•' by convention, 'v's scope extends over that of '•' implied by the juxtaposition of Q and R. Therefore 'P v QR' denotes 'P v $(Q \bullet R)$' rather than '$(P$ v $Q) \bullet R$'. The otherwise ambiguous expression '$P \supset Q$ v RS' contains three connectives: '⊃' which has precedence over 'v' which in turn has precedence over '•'. Therefore '$P \supset Q$ v RS'

[2] Note that there are no abbreviations is the object language, nor does the symbol 'v' even occur in the object language; hence our definition of 'P v Q' in the Syntax Language as identical to those *wffs* denoted by $\sim((\sim(P)) \bullet (\sim(Q)))$ in the object language.

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denotes '$P \supset [Q \text{ v } (R \bullet S)]$' rather than some other arrangement. And the otherwise ambiguous expression '$P \supset Q \equiv {\sim}Q \supset {\sim}P$' also contains three connectives: '$\equiv$' which has precedence over '$\supset$' which has precedence over '$\sim$'. Thus '$P \supset Q \equiv {\sim}Q \supset {\sim}P$' denotes '$[P \supset Q] \equiv [({\sim}Q) \supset ({\sim}P)]$' rather than some other arrangement. (p. 219)

Another convention is useful in disambiguating expressions where even order of precedence does not suffice. **Association to the left** requires that an expression's parts be grouped by parentheses *to the left*. Copi explains:

> [W]hen an expression contains two (or more) occurrences of the same connective, and their relative scopes within the expression are not otherwise indicated, the occurrence to the right should be understood to have the wider (or widest) scope. (*l.c.*)

E.g. The expression '$P \supset Q \supset P \supset P$' cannot be disambiguated by an appeal to order of precedence because all of its connectives have equal precedence. However, by associating to the left we can interpret it as denoting '$[(P \supset Q) \supset P] \supset P$'. The expression '$P \equiv Q \equiv PQ \text{ v } {\sim}P{\sim}Q$' can be partly disambiguated by applying the convention of order of precedence so that it denotes either '$[P \equiv Q] \equiv [(P \bullet Q) \text{ v } ({\sim}P \bullet {\sim}Q)]$' or $P \equiv \{Q \equiv [(P \bullet Q) \text{ v } ({\sim}P \bullet {\sim}Q)]\}$'. However, because the former is left associated, this is the interpretation supported by the convention. (p. 219 - 220)

The exercises on p. 220 provide practice with the abbreviations and conventions above by "undoing" them. Each of the expressions is of the Syntax Language and the task is to rewrite them in their unabbreviated form, complete with parentheses. By "unabbreviated" we take to mean "in terms of primitive symbols only". The first five solutions below are provided by Jon Ross. Unfortunately, his solutions can no longer be found in their original location.

1. $P \text{ v } {\sim}P$

   ${\sim}(({\sim}(P)) \bullet ({\sim}({\sim}P)))$

   ${\sim}(({\sim}(P)) \bullet ({\sim}({\sim}(P))))$

2. ${\sim}{\sim}P \supset P$

   $({\sim}{\sim}P) \supset P$

   ${\sim}({\sim}{\sim}P) \bullet ({\sim}(P)))$

   ${\sim}(({\sim}({\sim}(P))) \bullet ({\sim}(P)))$

3. $PQ \supset P$

   $(P \bullet Q) \supset P$

   ${\sim}((P \bullet Q) \bullet ({\sim}(P)))$

4. $PQ \vee R$

   $(P \bullet Q) \vee R$

   $\sim((P \bullet Q)) \bullet (\sim(R)))$

5. $P \supset (Q \supset R)$

   $P \supset (\sim((Q) \bullet (\sim(P))))$

   $\sim((P) \bullet (\sim(\sim((Q) \bullet (\sim(P))))))$

Functional Completeness

Whereas Copi has so far shown that R.S. is able to express several truth functions, in order to show that it is *functionally complete* he must show that it is able to express all possible truth functions. In other words, our semantical metalanguage must have a method of expressing all possible truth functions, from which we must be able to prove that all of these, or all their substitution instances, can be expressed in the object language as well, on its standard interpretation. In Critical Reasoning 05 we saw how truth tables provide a way of expressing all possible truth functions. Therefore truth tables and their notation can be freely introduced into our semantical metalanguage and can be used in discussing various semantic properties of R.S. on its standard interpretation. (p. 220)

Truth functions may have any number of arguments (or "independent variables" in the mathematical sense.) Thus $f(P)$ is a truth function if, and only, if its truth value is completely determined by the truth or falsehood of P. There are exactly four truth functions of a single argument that are *defined* by the following truth tables:

P	$f1(P)$
T	F
F	T

P	$f2(P)$
T	T
F	F

P	$f3(P)$
T	F
F	F

P	$f4(P)$
T	T
F	T

The functions $f1(P)$ to $f4(P)$ moreover are the *only* truth functions of a single argument, variously called 'unary', 'monadic' or 'singulary' functions. We can see that they can be expressed in R.S. under its normal or intended interpretations by first noting the normal or intended interpretations of $\sim P$ and PQ as defined by the following truth tables respectively:

P	$\sim P$
T	F
F	T

P	Q	PQ
T	T	T
T	F	F
F	T	F
F	F	F

(p. 220 - 221)

According to Copi, we can prove that R.S. is adequate to express $f1(P)$ to $f4(P)$ by actually going ahead and formulating them in R.S. We can see that $f2(P)$ is true when P is true and false when P is false and is therefore expressible in R.S. simply as P. $f1(P)$ on the other hand is false when P is true but true when P is false and is therefore expressible in R.S. as $\sim P$. $f3(P)$ meanwhile is always false, no matter what the truth value of P and is therefore expressible in R.S. as $\sim PP$. $f4(P)$ by contrast is always true, no matter what the truth value of P and is therefore expressible in R.S. as $\sim(\sim PP)$. Thus all singulary truth functions are expressible in R.S. (p. 221)

Truth functions that are of two arguments, also called 'binary' or 'dyadic', are defined by the following truth tables:

P	Q	$f1(P,Q)$
T	T	F
T	F	T
F	T	T
F	F	T

P	Q	$f2(P,Q)$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$f3(P,Q)$
T	T	T
T	F	T
F	T	F
F	F	T

P	Q	$f4(P,Q)$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$f5(P,Q)$
T	T	F
T	F	F
F	T	T
F	F	T

P	Q	$f6(P,Q)$
T	T	F
T	F	T
F	T	F
F	F	T

P	Q	$f7(P,Q)$
T	T	F
T	F	T
F	T	T
F	F	F

P	Q	$f8(P,Q)$
T	T	T
T	F	F
F	T	F
F	F	T

P	Q	$f9(P,Q)$
T	T	T
T	F	F
F	T	T
F	F	F

P	Q	$f10(P,Q)$
T	T	T
T	F	T
F	T	F
F	F	F

P	Q	$f11(P,Q)$
T	T	F
T	F	F
F	T	F
F	F	T

P	Q	$f12(P,Q)$
T	T	F
T	F	F
F	T	T
F	F	F

P	Q	$f13(P,Q)$
T	T	F
T	F	T
F	T	F
F	F	F

P	Q	$f14(P,Q)$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$f15(P,Q)$
T	T	F
T	F	F
F	T	F
F	F	F

P	Q	$f16(P,Q)$
T	T	T
T	F	T
F	T	T
F	F	T

Each of these binary functions, $f1(P,Q)$ to $f16(P,Q)$ defined by its corresponding truth table, is expressible in R.S. by means of the symbols $\sim$ and • alone. For example $f14(P,Q)$, which is true only when both P and Q are true together, can be expressed as *P* • *Q*, while $f1(P,Q)$, which is true only when P and Q are not true together, can be expressed as ~(*P* • *Q*). Copi leaves it as an exercise to express the remaining binary functions as *wffs* in R.S. (p. 221 - 222)

Jon Ross has provided solutions to all of these expressions, unfortunately they are no longer available at the original location. The first six are presented below.

$f1(P,Q)$ can be expressed as $\sim(P \bullet Q)$

$f2(P,Q)$ can be expressed as $\sim(P \bullet \sim Q)$

$f3(P,Q)$ can be expressed as $\sim(\sim P \bullet Q)$

$f4(P,Q)$ can be expressed as ~(~*P* • ~*Q*)

$f5(P, Q)$ can be expressed as $\sim(P \bullet Q) \bullet \sim(P \bullet \sim Q)$

$f6(P, Q)$ can be expressed as $\sim(P \bullet Q) \bullet \sim(\sim P \bullet Q)$

If you were able to correctly express these few, you should have no trouble with the rest.

Truth functions that are of three arguments, also called 'ternary' or 'triadic' *i.e.* $f1(P, Q, R)$; $f2(P, Q, R)$; … ; $f256(P, Q, R)$ are defined by 256 unique eight-row truth tables respectively. Copi sets it as an exercise to express any ten of them as a *wff* in R.S. Jon Ross has taken a systematic approach by setting out the first ten truth tables in a combined table. Writing out any ten random eight row truth tables as above would have been far more cumbersome and time consuming. We therefore follow his method in the table below.

P	Q	R	$f1$	$f2$	$f3$	$f4$	$f5$	$f6$	$f7$	$f8$	$f9$	$f10$
T	T	T	F	T	T	T	T	T	T	T	F	F
T	T	F	T	F	T	T	T	T	T	T	F	T
T	F	T	T	T	F	T	T	T	T	T	T	F
T	F	F	T	T	T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	F	T	T	T	T	T
F	T	F	T	T	T	T	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T	F	T	T	T
F	F	F	T	T	T	T	T	T	T	F	T	T

Here are the first five odd numbered expressions.

$f1(P, Q, R)$ can be expressed as $\sim(P \bullet Q \bullet R)$

$f3(P, Q, R)$ can be expressed as $\sim(P \bullet \sim Q \bullet R)$

$f5(P, Q, R)$ can be expressed as $\sim(\sim P \bullet Q \bullet R)$

$f7(P, Q, R)$ can be expressed as $\sim(\sim P \bullet \sim Q \bullet R)$

$f9(P, Q, R)$ can be expressed as $\sim(P \bullet Q \bullet R) \bullet \sim(P \bullet Q \bullet \sim R)$

According to Copi, in order to prove the functional completeness of R.S. it is necessary to show that *any* truth function of *any* number of arguments is expressible as a *wff* of R.S. by means of means of the symbols $\sim$ and $\bullet$ alone. Any n-adic truth function $f(P_1;\ P_2;\ \ldots;P_{n-1};\ P_n)$ is completely specified by a truth table of n initial columns and 2^n rows and such that a 'T' or an 'F' occupies each of the 2^n places of the last coulomb, as follows:

	P_1	P_2	…	P_{n-1}	P_n	$f(P_1;\ P_2;\ \ldots;P_{n-1};\ P_n)$
Row 1:	T	T	…	T	T	
Row 2:	T	T	…	T	F	
…………						
Row 2^n - 1	F	F	…	F	T	
Row 2^n	F	F	…	F	F	

As in any truth table, the coulombs to the left are filled in such a way that every permutation of T's and F's is represented. It pays to do this systematically (as described in Critical Reasoning 05) so that there are no omissions or duplications. Jon Ross has done just that in his truth table for three permutations above, however there is no theoretical limit on the number of permutations that may be required depending on the number of arguments involved. The final coulomb to the right, $f(P_1;\ P_2;\ \dots;P_{n-1};\ P_n)$ must have either an F in one of its rows or an F in more than one of its rows or no F's in any of its rows. In either case the truth function can be represented by a *wff* in R.S. (p. 222)

Consider the following three cases presented by Copi which exhaust the number possibilities above:

Case 1: $f(P_1;\ P_2;\ \dots;P_{n-1};\ P_n)$ has an F on one, and only one, row of its truth table. If the F is in the first row then the truth function can be represented by $\sim(P_1 \bullet P_2 \bullet\ \dots \bullet P_{n-1} \bullet P_n)$ as a *wff* in R.S. If the F is in the second row of the truth table then the truth function can be represented by $\sim(P_1 \bullet P_2 \bullet\ \dots \bullet P_{n-1} \bullet \sim P_n)$ as a *wff* in R.S. If the F is on the ith row of the truth table then all 2^n distinct truth functions can be represented by

$$S_1: \sim(P_1 \bullet P_2 \bullet\ \dots \bullet P_{n-1} \bullet P_n)$$

$$S_2: \sim(P_1 \bullet P_2 \bullet\ \dots \bullet P_{n-1} \bullet \sim P_n)$$

…………………………………………………………………….

$$S_{2^n-1}: \sim(\sim P_1 \bullet \sim P_2 \bullet\ \dots \bullet \sim P_{n-1} \bullet P_n)$$

$$S_{2^n-1}: \sim(\sim P_1 \bullet \sim P_2 \bullet\ \dots \bullet \sim P_{n-1} \bullet \sim P_n)$$

as *wffs* in R.S.

Case 2: $f(P_1, P_2, \dots\ P_{n-1},\ P_n)$ has an F on more than one row of its truth table. If the F's are on the $k(1 < k \leq 2^n)$ rows i_1, i_2, … i_k of the truth table then the truth function can be represented by

$$S_{i_1} \bullet S_{i_2} \bullet \dots \bullet S_{i_k}$$

as *wffs* in R.S.

Case 3: $f(P_1, P_2, \dots\ P_{n-1},\ P_n)$ has no F's on any row of the truth table, in which case the truth function is a tautology and can be represented by

$$\sim(\sim P_1 \bullet P_1 \bullet P_2 \bullet\ \dots \bullet P_{n-1} \bullet P_n)$$

as a *wff* in R.S. (p. 222 - 223)

Since these cases exhaust all the possibilities above, Copi has shown that it is possible to express any truth function of any number of arguments as a *wff* in R.S. Note however that this is not a theorem of R.S. but a theorem *about* R.S. in our semantical metalanguage, therefore it may be stated as

Metatheorem I. *R.S. is functionally complete*

If such a proof were not sufficient, a more rigorous proof requires mathematical induction, of which there are two types. According to Copi, 'weak induction' refers to the type more commonly used in mathematical induction. This has the form of

$$\begin{array}{l} f(1) \\ \underline{\text{for any arbitrary } m\text{, if } f(m) \text{ then } f(m+1)} \\ \therefore f(m) \text{ for every } m \end{array}$$

Metaphorically, mathematical induction can be thought of as proving that we can climb a ladder as high as we like. First we prove that we can climb onto the bottom rung (the base) and then that from each rung we can climb onto the next (the induction.) (Graham, *et al*. 1994, p. 3)

The form above is commonly used to prove theorems in elementary algebra, for which Copi provides an example: The sum of the first n odd integers is equal to n^2. We can see that the first premise, which Copi calls the 'α-case' is true by the trivial equation 1 = 1^2. The second premise, which Copi calls the 'β-case', can be established by assuming that $f(m)$ is true for any arbitrary integer m and from this deriving that $f(m + 1)$ is true. In this example, the β-case assumption is

$$1 + 3 + 5 + \cdots + (2m - 1) = m^2$$

By adding ($2m$ + 1) to both sides we obtain:

$$1 + 3 + 5 + \cdots + (2m - 1) + (2m + 1) = m^2 + (2m + 1)$$

and then by regrouping and factorising,

$$1 + 3 + 5 + \cdots + (2m - 1) + [2(m + 1) - 1] = (m + 1)^2$$

which shows that the sum of the first m odd integers is m^2 and then that the first $m + 1$ odd integers is $(m + 1)^2$, which is the β-case. From both cases we infer, by weak induction, the desired conclusion that "for every m, the sum of the first m odd integers is equal to m^2". (p. 223 - 224)

According to Copi, weak induction can be thought of as summarising an unending sequence of *modus ponens* arguments of the form

$$\begin{array}{l} f(1) \\ \underline{f(1) \supset f(2)}; \\ \therefore f(2) \end{array} \qquad \begin{array}{l} f(2) \\ \underline{f(2) \supset f(3)}; \ldots; \\ \therefore f(3) \end{array} \qquad \begin{array}{l} f(m) \\ \underline{f(m) \supset f(m+1)}; \ldots \\ \therefore f(m+1) \end{array}$$

To prove Metatheorem I. by week induction we can state both cases as follows:

α-case: As shown at the beginning of this section, any truth function of a single argument $f(P)$ can be expressed as a *wff* in R.S.

β-case: If we assume that any truth function of k arguments $(P_1;\ P_2; \ldots; P_k)$ can be expressed as a *wff* in R.S., we must then prove that any truth function of k + 1 arguments $g(P_1;\ P_2; \ldots; P_k;\ P_{k+1})$ can also be expressed as a *wff* in R.S. If we let $g(P_1;\ P_2; \ldots; P_k;\ P_{k+1})$ be the truth function of any k + 1 arguments, $P_1;\ P_2; \ldots; P_k;\ P_{k+1}$, then the former is defined by the truth table, over page:

P_1	P_2	…	P_k	P_{k+1}	$g(P_1; P_2; \ldots; P_k; P_{k+1})$
T	T		T	T	V_1
T	T		T	F	V_2
……					
T	F		F	T	V_{2^k-1}
T	F		F	F	V_{2^k}
F	T		T	T	V_{2^k+1}
F	T		T	F	V_{2^k+2}
……					
F	F		F	T	$V_{2^{k+1}-1}$
F	F		F	F	$V_{2^{k+1}}$

where each V_i ($1 \leq i \leq 2^{k+1}$) is a truth valve, T or an F.

Now a truth function $f_1(P_2; P_3 \ldots; P_k; P_{k+1})$ of k arguments from P_2 to P_{k+1} can be defined by the first 2^k rows of the above truth table with its first column deleted. Thus:

P_2	P_3	…	P_k	P_{k+1}	$f_1(P_2; P_3; \ldots; P_k; P_{k+1})$
T	T		T	T	V_1
T	T		T	F	V_2
……					
F	F		F	T	V_{2^k-1}
F	F		F	F	V_{2^k}

which, by the β-case assumption above, can be expressed as a *wff* in R.S - call it S_1. Similarly a truth function $f_2(P_2; P_3 \ldots; P_k; P_{k+1})$ of k arguments from P_2 to P_{k+1} can be defined by the *last* 2^k rows of the first truth table above with its first column deleted. See over page.

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P_2	P_3	…	P_k	P_{k+1}	$f_2(P_2; P_3;\ \dots; P_k;\ P_{k+1})$
T	T		T	T	V_{2^k+1}
T	T		T	F	V_{2^k+2}
…………					
F	F		F	T	$\mathrm{V}_{2^{k+1}-1}$
F	F		F	F	$\mathrm{V}_{2^{k+1}}$

which, again by the β-case assumption above, can be expressed as a *wff* in R.S. - call it S_2.

According to Copi, the truth table that defines the *wff* $P_1 \bullet S_1$ has the same entries as the original table in its first 2^k rows and only F's in its *last* 2^k rows. Please verify! On the other hand the truth table that defines the *wff* $\sim P_1 \bullet S_2$ has only F's in its first 2^k rows and the same entries as the original table in its *last* 2^k rows. Again the reader may wish to verify this. Therefore, the truth table that defines the *wff* $(P_1 \bullet S_1)$ v $(\sim P_1 \bullet S_2)$ has exactly the same entries as those of the original truth table. Hence, the arbitrary truth function, $g(P_1; P_2;\ \dots; P_k;\ P_{k+1})$ of k + 1 arguments can be expressed as a *wff* in R.S. as $(P_1 \bullet S_1)$ v $(\sim P_1 \bullet S_2)$ or as $\sim(\sim(P_1 \bullet S_1) \bullet \sim(\sim P_1 \bullet S_2)$, which establishes the β-case.

Together with the α-case, metatheorem I follows by weak induction. (p. 224 - 225)

Strong induction is not so called because it proofs are some 'stronger' than 'weak' induction. In fact they are equivalent, both relying on summarising an unending sequence of *modus ponens* arguments. However, if strong induction is valid then weak induction is also valid as a special case. Strong induction has the form:

$f(1)$
for any arbitrary m, if $f(k)$ for every $k < m$ then $f(m)$
$\therefore f(m)$ for every m

The sequence of *modus ponens* arguments relied on by strong induction are of the form:

$$\frac{\begin{array}{l} f(1) \\ f(1) \supset f(2); \end{array}}{\therefore f(2)} \quad \frac{\begin{array}{l} f(1) \bullet f(2) \\ f(1) \bullet f(2) \supset f(3) \end{array}}{\therefore f(3)};\ \dots;\ \frac{\begin{array}{l} f(1) \bullet f(2) \bullet \dots \bullet f(m-1) \\ f(1) \bullet f(2) \bullet \dots \bullet f(m-1) \supset f(m) \end{array}}{\therefore f(m)};\ \dots$$

Copi illustrates the use of strong induction by proving that a logistic system based only on the propositional symbols P, Q, R, S… and the logical connectives • and v *cannot* be functionally complete. This is done by proving that no *wff* of such a system can express a truth function that is true when all of its arguments are false. (p. 225 - 226)

α-case: If $g(P, Q, R, \dots)$ contains only one symbol then to be a *wff* it must be either P alone or Q alone or R or… Where the arguments P, Q, R… are all false, $g(P, Q, R, \dots)$ will also be false because it is one of them. Hence any *wff* of such a system that contains exactly one symbol cannot be true when all its arguments are false.

β-case: We assume that any *wff* $g(P,Q,R,\dots)$ containing fewer than m symbols cannot be true when all its arguments are false. Consider any formula $g(P,Q,R,\dots)$ that contains exactly m symbols where $m > 1$. If $g(P,Q,R,\dots)$ is a *wff* then it must be either

$$g_1(P,Q,R,\dots) \bullet g_2(P,Q,R,\dots) \quad \text{or} \quad g_1(P,Q,R,\dots) \vee g_2(P,Q,R,\dots)$$

where $g_1(P,Q,R,\dots)$ and $g_2(P,Q,R,\dots)$ are themselves *wffs* containing fewer than m symbols. However by the β-case assumption when all the arguments P, Q, R… are all false $g_1(P,Q,R,\dots)$ and $g_2(P,Q,R,\dots)$ will both be false too. Furthermore both *their* conjunction or *their* disjunction will also both be false. Hence, $g(P,Q,R,\dots)$ will be false, which establishes the β-case.

Together with the α-case, it follows by strong induction that no *wff* based only on the propositional symbols P,Q,R,S… and the logical connectives • and v can be true when all its arguments are false and hence it cannot be functionally complete. (p. 226)

In the exercises that follow Copi requires the reader to prove the functional completeness or incompleteness of various logistic systems based system based only on the propositional symbols P, Q, R, S… , parentheses and the logical connectives presented in each case. We are not told by which method we should construct our proofs, indeed those provided by Jon Ross are all different but model solutions. We reproduce nos. 3 and 4 below.

3. $\supset$ and $\sim$

$S(\supset,\sim)$ is functionally complete because it already contains the '$\sim$' of R.S. and because the '•' of R.S. is definable in $S(\supset,\sim)$ because $(P \bullet Q) \equiv \sim(P \supset \sim Q)$. Thus, any truth function that is expressible in R.S. is expressible in $S(\supset,\sim)$

4. $\supset$ and v

$S(\supset,\text{v})$ is functionally incomplete because there is no *wff* in $S(\supset,\text{v})$ that expresses a truth function that is false when all of its arguments are true. The proof proceeds by strong induction. Consider the *wff* $g(P,Q,R,\dots)$ of $S(\supset,\text{v})$:

α-case: If $g(P,Q,R,\dots)$ contains just one symbol then then to be a *wff* it must be either P alone or Q alone or R or… If each of these is true, then $g(P,Q,R,\dots)$ cannot be false.

β-case: We assume that any *wff* $g(P,Q,R,\dots)$ containing fewer than m symbols cannot be false when all its arguments are true. Consider any formula $g(P,Q,R,\dots)$ that contains exactly m symbols where $m > 1$. If $g(P,Q,R,\dots)$ is a *wff* then it must be either

$$g_1(P,Q,R,\dots) \supset g_2(P,Q,R,\dots) \quad \text{or} \quad g_1(P,Q,R,\dots) \vee g_2(P,Q,R,\dots)$$

where $g_1(P,Q,R,\dots)$ and $g_2(P,Q,R,\dots)$ are themselves *wffs* of $S(\supset,\text{v})$ containing fewer than m symbols. However by the β-case assumption when all the arguments P, Q, R… are all true then $g_1(P,Q,R,\dots)$ and $g_2(P,Q,R,\dots)$ will both be true also. Since both $T \supset T$ and $T \vee T$ are both true, $g(P,Q,R,\dots)$ cannot be false when all its arguments are true. Therefore no *wff* in $S(\supset,\text{v})$ that expresses a truth function can be false when all of its arguments are true. Hence $S(\supset,\text{v})$ is functionally incomplete.

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Axioms and Demonstrations

The rules and proofs of theorems within any formal system can be greatly simplified by assuming infinitely many *wffs* as axioms or postulates. As Copi points out, we obviously cannot write out an infinite list of axioms; however we can use our Syntax Language to specify which *wffs* of R.S. qualify as axioms and which do not, so long as there is an effective process for deciding among them. The list of axioms of R.S. expressed in their syntactical formulation is as follows;

Axiom 1. $P \supset (P \bullet P)$

Axiom 2. $(P \bullet Q) \supset P$

Axiom 3. $(P \supset Q) \supset [\sim(Q \bullet R) \supset \sim(R \bullet P)]$

Each of these axioms represents or designates an infinite list of *wffs* in the object language. Thus Axiom 1 designates all of the following:

$\sim((\boldsymbol{A}) \bullet (\sim((\boldsymbol{A}) \bullet (\boldsymbol{A}))))$

$\sim((\boldsymbol{B}) \bullet (\sim((\boldsymbol{B}) \bullet (\boldsymbol{B}))))$

$\sim((\boldsymbol{C}) \bullet (\sim((\boldsymbol{C}) \bullet (\boldsymbol{C}))))$

$\sim((\boldsymbol{D}) \bullet (\sim((\boldsymbol{D}) \bullet (\boldsymbol{D}))))$

$\sim((\boldsymbol{A_1}) \bullet (\sim((\boldsymbol{A_1}) \bullet (\boldsymbol{A_1}))))$

…………………………………………………………….

$\sim((\sim(\boldsymbol{A})) \bullet (\sim((\sim(\boldsymbol{A})) \bullet (\sim(\boldsymbol{A})))))$

$\sim((\sim(\boldsymbol{B})) \bullet (\sim((\sim(\boldsymbol{B})) \bullet (\sim(\boldsymbol{B})))))$

……………………………………………………………………………

$\sim(((\boldsymbol{A}) \bullet (\boldsymbol{D})) \bullet (\sim(((\boldsymbol{A}) \bullet (\boldsymbol{D})) \bullet ((\boldsymbol{A}) \bullet (\boldsymbol{D})))))$

$\sim(((\boldsymbol{A_3}) \bullet (\boldsymbol{B_7})) \bullet (\sim(((\boldsymbol{A_3}) \bullet (\boldsymbol{B_7})) \bullet ((\boldsymbol{A_3}) \bullet (\boldsymbol{B_7}))))$

…………………………………………………………………………………………..

and infinitely more *wffs* of R.S. that are of the same pattern. Of the three patterns listed in their syntactical formulations as axioms, every *wff* of R.S. in the object language that exemplifies one of these patterns is accounted an axiom of R.S. (p. 227 - 228)

We may regard the axioms above as reasonable assumptions because, on the intended interpretation of ~ and •, they are tautologies. One of the intended purposes for our logistic system, upon interpretation, is to formulate arguments. Arguments, recall, consist of a list of premises and conclusions expressed as statements. Corresponding to such arguments are a list of *wffs* in our logistic system, the last of which in each case expresses the conclusion. Clearly it is desirable to have some criteria by which to decide which of these sequences of *wffs* correspond to valid arguments, when interpreted normally, and which do not. (p. 228)

In Critical Reasoning 07 we introduced the method of deduction by which a formal proof could be given to demonstrate that an argument is valid. Then we relied upon a number of elementary rules of rules of inference that were assumed to be valid and, later in the same study unit, ten logically equivalent rules of replacement in formulating such proofs. These were supplemented the methods of conditional proof and indirect poof in Critical Reasoning 09. Some, like Jon Ross above, have reservations about the latter for metaphysical reasons, and there may be others. Therefore when we come to make assumptions about valid rules of inference for our logistic system, we should keep them to a bare minimum; however we cannot make no assumptions, for then there will be no inference legitimised within it at all. Copi therefore assumes only one rule of inference for the validation of all arguments and for the poof of all theorems within R.S., namely that of *Modus Ponens*. Stated in the Syntax Language, we have

Rule 1. From P and $P \supset Q$ infer Q (*l.c.*)

Any argument in R.S. that has the above form is legitimated as valid by this assumption. Rule 1 (abbreviated R1) by itself only legitimates special kinds of arguments having two premises. R.S. however it is able to express all formally valid arguments that can be verified by truth tables, including those with multiple premises; therefore it is desirable to have a method by which such arguments can be validated by repeated uses of R1. Copi introduces such a method on p. 229.

Generally, a logical **demonstration** is a discourse proceeding from axioms to one or more theorems, according to specified deductive system**.** In this specific context, a demonstration of validity (of an argument) that has premises P_1; P_2; … P_n and a conclusion Q is defined as a sequence of *wffs* S_1; S_2 … S_k such that every S_i is either one of the premises P_1; P_2; … P_n, or one of the axioms of R.S., or follows from the preceding two S's by R1 and such that S_k is Q. An argument then is regarded as valid if, and only if, there exists a *demonstration* of its validity. Copi introduces a new symbol into the Syntax Language to represent just this. The symbol '⊢' (read 'yields') may be used as follows so that

$$P_1; P_2; \ldots P_n \vdash Q$$

asserts that there is a demonstration of validity of the argument having premises P_1; P_2; … P_n and a conclusion Q. Consider Copi's example of all the infinitely many arguments denoted by

$$\begin{array}{l} P \supset Q \\ \underline{\sim(QR)} \\ \therefore \sim(RP) \end{array}$$

All of them that have this form are valid because $P \supset Q; \sim(QR) \vdash \sim(RP)$. The demonstration proceeds according to the following sequence of S's:

S_1: $(P \supset Q) \supset [\sim(QR) \supset \sim(RP)]$ which is Axiom 3

S_2: $P \supset Q$ which is the first premise

S_3: $\sim(QR) \supset \sim(RP)$ which follows from S_1 and S_2 by R1

S_4: $\sim(QR)$ which is the second premise

S_5: $\sim(RP)$ which follows from S_3 and S_4 by R1. (p. 299)

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In general, any such sequence of S's is a may be called a demonstration that P_1; P_2; … $P_n \vdash Q$. Each S_i is called a **line** of the demonstration. According to Copi, when a demonstration is given for the validity of all arguments of a R.S. of a specific form, any instance of that form may be regarded as validated by that demonstration and the general form of that argument may be regarded as a **derived rule of inference** *i.e.* a rule of inference rule not given as part of the derivation system but which constitutes an abbreviation using a previously proved theorem (Wikibooks). When the axioms of a system are the only premises of a valid argument, then its conclusion is a **theorem** of that system. To indicate that a formula Q is a theorem we write '$\vdash Q$', which is defined as a demonstration of Q which is sequence of *wffs* S_1; S_2 … S_k such that every S_i is either an axiom of R.S. or follows from the preceding two S's by R1 and such that S_k is Q. (p. 229 - 230)

Copi points out that demonstration, either of a theorem or of the validity of an argument, proceeds by an **effective method** or **effective procedure**. Given any sequence of S's, however long, it can be decided by a purely mechanical procedure, in a finite number of steps, whether or not it is a demonstration. Similarly, it can be decided effectively whether any given S is an axiom and whether or not it serves as a premise when considering an argument. If some S, say S_j, is neither an axiom nor a premise, then it can be decided in a finite number of steps whether or not the two preceding lines before S_j are S_i and $S_i \supset S_j$, in which case S_j will follow by R1. (p. 230)

With the above definition of a theorem, the three axioms and one rule of R.S. can be regarded as parts of a "symbolic machine" that generates *wffs*. Any axiom on its own generates an infinite number of *wffs* and by applying R1 to them repeatedly, infinitely more *wffs* are produced as theorems. But will this system be consistent and are all the theorems so produced tautologies? According to Copi, "A logistic system that is a propositional calculus (such as R.S.) will be called **analytic**, if and only if, all of its theorems become tautologies on their normal interpretations. R.S. is analytic, by this definition, provided that $\vdash P$ implies that P is tautologous." (*l.c.*)

We can now state this as metatheorem II.

Metatheorem II. *R.S. is analytic* (*i.e. if* $\vdash P$ *then* P *is a tautology*)

Copi's proof involves the use of strong induction on the number of times R1 is used in the demonstration that $\vdash P$. (*l.c.*)

α-case: If P results from a single use of R1 then it must be applied to the axioms. We can manually verify that the axioms are indeed all tautologies by constructing truth tables for them. Hence P results from the application of R1, once, to S and $S \supset P$ where S and $S \supset P$ are both tautologies. Therefore P must be a tautology. If it were not, then there would be an F in at least one row j of its truth table. But since S is a tautology, all of its truth table rows have a T on them, including a T on row j. But if there were a T on row j of S and a an F row j of P, then $S \supset P$ would have an F on one its rows, which is a contradiction, since it is a tautology. Therefore Metatheorem II is true for m = 1 use of R1.

β-case: We assume that Metatheorem II is true for any number of $k < m$ uses of R1. In other words, if S is a theorem whose demonstration requires fewer than m uses of R1, then it is a tautology. For any P that has a demonstration that requires exactly m uses of R1, either P is an axiom, in which case it is already a tautology, or P follows from the two previous lines S and $S \supset P$, by the m^{th} use of R1. For the previous line S, either it itself is an axiom, in which case it is already a tautology, or it followed by $k < m$ uses of R1 in which case it is a tautology by the β-case assumption. By the same reasoning, the prior line $S \supset P$ must also be a tautology. Now since S and $S \supset P$ are both tautologies, P must be also be a tautology by the α-case

Together with the α-case, it follows by strong induction that if $\vdash P$ is the case, by any number of uses of R1, then P must be a tautology, hence R.S. is analytic.

Since R.S. has been shown to be analytic, it follows immediately as a corollary of Metatheorem II that it is consistent, the proof of which involves the Post criterion of consistency introduced in Critical Reasoning 16. We can now state the corollary followed by its proof.

Corollary: *R.S. is consistent*

Proof: $P \bullet \sim P$ is a *wff* of R.S. but it is not a tautology, therefore by Metatheorem II it not a theorem of R.S. either. According to the Post criterion of consistency, since R.S. contains a formula that is not provable as a theorem, R.S. must therefore be consistent. (p. 231)

Copi describes the **deductive completeness** of a logistic system as converse to its analyticity. Whereas a system is analytic when all its theorems are tautologies, it may be called *deductively complete* when all its tautologies are provable as theorems within the system. Proving the latter requires a diversion into the independence of the axioms in the next section and the development of some theorems of the system in the penultimate section. (*l.c.*)

Independence of the Axioms

According to Wikipedia: Axiom independence, "An axiom P is independent if there are no other axioms Q such that Q implies P… If the original axioms Q are not consistent, then no new axiom is independent. If they are consistent, then P can be shown independent of them if adding P to them, or adding the negation of P, both yield consistent sets of axioms."

For Copi, "A set of axioms is said to be independent if each of them is independent of the others, that is, if none of them can be derived as theorems from the others." So far this amounts to the same definition as above; however his criteria for proving that an axiom is independent differ in practice. Thus, "To prove the independence of each axiom, it suffices to find a characteristic of *wffs* of the system such that:

1. The axiom to be proved independent lacks that characteristic.
2. All other axioms have that characteristic.
3. The characteristic in question is hereditary with respect to the rules of inference of the system." (p. 231)

Copi uses **hereditary** here to mean a characteristic, with respect to a set of rules of inference that, if and only if, it belongs to one or more formulae, it also belongs to every formula deduced from them by those rules of inference. The characteristic of being tautologous, for example, is hereditary in this respect; however it cannot be used to prove an axiom of R.S. independent because all axioms of R.S. have the characteristic of being tautologous. (p. 231 - 232)

To prove the independence of the axioms of R.S. Copi uses a model containing three elements by which to interpret the symbols and *wffs* of R.S. The 'values' 0, 1 and 2 are used analogously to the truth values T and F in truth tables. Every propositional symbol has one of these values assigned to it and every *wff* that is not a propositional symbol has one of the values assigned to it according to the following tables:

P	$\sim P$	P	Q	$P \bullet Q$
0	2	0	0	0
1	1	0	1	1
2	0	0	2	2
		1	0	1
		1	1	2
		1	2	2
		2	0	2
		2	1	2
		2	2	2

The symbol '⊃'is defined by

$$P \supset Q = df \sim(P \bullet \sim Q)$$

Therefore from this definition we have the table:

P	Q	$P \supset Q$
0	0	0
0	1	1
0	2	2
1	0	0
1	1	0
1	2	1
2	0	0
2	1	0
2	2	0

The characteristic of *wffs* that Copi identifies as belonging to Axiom 2 and Axiom 3 of R.S. is that of having a value of 0 on the model regardless of the assigned values of its component propositional symbols. We can then create and fill in tables for these axioms analogous to the way that we filled in truth tables for compound formulae in Critical Reasoning 05. Thus for Axiom 2: $(P \bullet Q) \supset P$ we have the table, over page:

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$(P$	$\bullet$	$Q)$	$\supset$	P
0	0	0	0	0
0	1	1	0	0
0	2	2	0	0
1	1	0	0	1
1	2	1	0	1
1	2	2	0	1
2	2	0	0	2
2	2	1	0	2
2	2	2	0	2

And for Axiom 3: $(P \supset Q) \supset [\sim(Q \bullet R) \supset \sim(R \bullet P)]$ we have the following table. Note that again there are only 0's under the operator with the largest scope.

$(P$	$\supset$	$Q)$	$\supset$	$[\sim$	$(Q$	$\bullet$	$R)$	$\supset$	$\sim$	$(R$	$\bullet$	$P)]$
0	0	0	0	2	0	0	0	0	2	0	0	0
0	0	0	0	1	0	1	1	0	1	1	1	0
0	0	0	0	0	0	2	2	0	0	2	2	0
0	1	1	0	1	1	1	0	1	2	0	0	0
0	1	1	0	0	1	2	1	1	1	1	1	0
0	1	1	0	0	1	2	2	0	0	2	2	0
0	2	2	0	0	2	2	0	2	2	0	0	0
0	2	2	0	0	2	2	1	1	1	1	1	0
0	2	2	0	0	2	2	2	0	0	2	2	0
1	0	0	0	2	0	0	0	0	1	0	2	1
1	0	0	0	1	0	1	1	0	0	1	1	1
1	0	0	0	0	0	2	2	0	0	2	2	1
1	0	1	0	1	1	1	0	0	1	0	2	1
1	0	1	0	0	1	2	1	0	0	1	1	1
1	0	1	0	0	1	2	2	0	0	2	2	1
1	1	2	0	0	2	2	0	1	1	0	2	1
1	1	2	0	0	2	2	1	0	0	1	2	1
1	1	2	0	0	2	2	2	0	0	2	2	1
0	0	0	0	2	0	0	0	0	0	0	2	0
0	0	0	0	1	0	1	1	0	0	1	2	0
0	0	0	0	0	0	2	2	0	0	2	2	0
0	0	1	0	1	1	1	0	0	0	0	2	0
0	0	1	0	0	1	2	1	0	0	1	2	0
0	0	1	0	0	1	2	2	0	0	2	2	0
0	0	2	0	0	2	2	0	0	0	0	2	0
0	0	2	0	0	2	2	1	0	0	1	2	0
0	0	2	0	0	2	2	2	0	0	2	2	0

The characteristic of having a value of 0 can be seen to be hereditary with respect to R1 of R.S. because according to the table for $\supset$ above, the only line on which both P and $P \supset Q$ have the value 0, Q also has the value of 0. Therefore if this characteristic belongs to one or more *wffs* it also belongs to every *wff* deduced from them by R1. Finally, the characteristic of a *wff* having a 0 on the model does not apply to Axiom 1: $P \supset (P \bullet P)$. When P is assigned the value of 1 (rather than 0) we

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get $1 \supset (1 \bullet 1)$ and thence $1 \supset 2$ which is 1. Hence Axiom 1 is independent for lacking the characteristic 0 shared by the other axioms. (p. 233 - 234)

Before proceeding to prove the independence of Axioms 2 and 3, Copi makes some general remarks about choosing a model to use and deciding what values to assign in making a table for the primitive symbols of the system as well as deciding which element(s) to designate as the hereditary characteristic. In sort, there is no effective (mechanical) way of choosing, except by being guided by economy, consistency as well as trial and error. (p. 234)

To prove the independence of Axiom 2 Copi uses the same three-element model as above, with the same table for the definition for $\sim P$ but with a difference for that of $P \bullet Q$ and derivatively for $P \supset Q$, thus:

P	Q	$P \bullet Q$	$P \supset Q$
0	0	0	0
0	1	0	2
0	2	2	2
1	0	0	0
1	1	0	2
1	2	2	2
2	0	2	0
2	1	2	0
2	2	2	0

When we fill in the tables for the *wffs* corresponding to Axioms 1 and 3 on the model according to the above definition we get the same characteristic value of 0 underneath the operator with the widest scope, regardless of the values assigned to its components, thus:

P	$\supset$	$(P$	$\bullet$	P
0	0	0	0	0
1	0	1	0	1
2	0	2	2	2

and over page,

$(P$	$\supset$	$Q)$	$\supset$	$[\sim$	$(Q$	$\bullet$	$R)$	$\supset$	$\sim$	$(R$	$\bullet$	$P)]$
0	0	0	0	2	0	0	0	0	2	0	0	0
0	0	0	0	2	0	0	1	0	2	1	0	0
0	0	0	0	0	0	2	2	0	0	2	2	0
0	2	1	0	2	1	0	0	0	2	0	0	0
0	2	1	0	2	1	0	1	0	2	1	0	0
0	2	1	0	0	1	2	2	0	0	2	2	0
0	2	2	0	0	2	2	0	2	2	0	0	0
0	2	2	0	0	2	2	1	2	2	1	0	0
0	2	2	0	0	2	2	2	0	0	2	2	0
1	0	0	0	2	0	0	0	0	2	0	0	1
1	0	0	0	2	0	0	1	0	2	1	0	1
1	0	0	0	0	0	2	2	0	0	2	2	1
1	2	1	0	2	1	0	0	0	2	0	0	1
1	2	1	0	2	1	0	1	0	2	1	0	1
1	2	1	0	0	1	2	2	0	0	2	2	1
1	2	2	0	0	2	2	0	2	2	0	0	1
1	2	2	0	0	2	2	1	2	2	1	0	1
1	2	2	0	0	2	2	2	0	0	2	2	1
2	0	0	0	2	0	0	0	0	0	0	2	2
2	0	0	0	2	0	0	1	0	0	1	2	2
2	0	0	0	0	0	2	2	0	0	2	2	2
2	0	1	0	2	1	0	0	0	0	0	2	2
2	0	1	0	2	1	0	1	0	0	1	2	2
2	0	1	0	0	1	2	2	0	0	2	2	2
2	0	2	0	0	2	2	0	0	0	0	2	2
2	0	2	0	0	2	2	1	0	0	1	2	2
2	0	2	0	0	2	2	2	0	0	2	2	2

Again, the characteristic of having a value of 0 can be seen to be hereditary with respect to R1 of R.S. because, according to the table for ⊃ above, the only line on which both P and $P \supset Q$ have the value 0, Q also has the value of 0. Therefore if this characteristic belongs to one or more *wffs* it also belongs to every *wff* deduced from them by R1. Now, the characteristic of a *wff* having a 0 on the model does not apply to Axiom 2: $(P \bullet Q) \supset P$. When P and Q are assigned the value of 1 (rather than 0) we get $(1 \bullet 1) \supset 1$ and thence $0 \supset 1$ which is 2. Hence Axiom 2 is independent for lacking the characteristic 0 shared by the other axioms. (p. 234 - 236)

To prove the independence of Axiom 3 Copi uses the same three-element model as above, with the same table for the definition for $\sim P$ but with a difference for that of $P \bullet Q$ and derivatively for $P \supset Q$, thus over page:

P	Q	$P \bullet Q$	$P \supset Q$
0	0	0	0
0	1	1	1
0	2	2	2
1	2	2	0
1	1	2	0
1	2	2	0
2	0	2	0
2	1	2	0
2	2	2	0

When we fill in the tables for the *wffs* corresponding to Axioms 1 and 2 on the model according to the above definition we get the same characteristic value of 0 underneath the operator with the widest scope, regardless of the values assigned to its components, thus:

P	$\supset$	$(P$	$\bullet$	$P)$
0	0	0	0	0
1	0	1	2	1
2	0	2	2	2

and

$(P$	$\bullet$	$Q)$	$\supset$	P
0	0	0	0	0
0	1	1	0	0
0	2	2	0	0
1	2	0	0	1
1	2	1	0	1
1	2	2	0	1
2	2	0	0	2
2	2	1	0	2
2	2	2	0	2

Yet again, the characteristic of having a value of 0 can be seen to be hereditary with respect to R1 of R.S. because according to the table for ⊃ above, the only line on which both P and $P \supset Q$ have the value 0, Q also has the value of 0. Therefore if this characteristic belongs to one or more *wffs* it also belongs to every *wff* deduced from them by R1. But, the characteristic of a *wff* having a 0 on the model does not apply to Axiom 3: $(P \supset Q) \supset [\sim(Q \bullet R) \supset \sim(R \bullet P)]$. When P and Q are assigned the value of 1 and R is assigned the value of 0 we get $(1 \supset 1) \supset [\sim(1 \bullet 0) \supset \sim(0 \bullet 1)]$ and thence $0 \supset [\sim 2 \supset \sim 1]$ followed by $0 \supset [0 \supset 1]$ and then $0 \supset 1$ and finally 1. Hence Axiom 3 is independent for lacking the characteristic 0 shared by the other axioms. (p. 236 - 237)

Copi provides four further sets of axioms for propositional calculi and leaves it as an exercise to prove the independence of each axiom. (p. 237) The interested reader may wish to attempt these, however they are rather time consuming, although Jon Ross provided abridged solutions, showing only the relevant portions of the required tables. However it is no longer available at its initial location. What is important is that we understand *how* Copi has proved the independence of the axioms for R.S. and *how* that task can be generalised to other logistic systems.

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Development of the Calculus

Copi reminds us that, motivated by the desire for rigor in the development of derived rules and theorems of the object language, all symbols are regarded as completely uninterpreted. When however we come to interpret the formulae $\sim RP$ and $P\sim R$ normally, we see that they are logically equivalent. However they cannot be regarded as logically equivalent *wffs* in its development phase, nor can $\sim RP \equiv P\sim R$ be accepted as a theorem of R.S. until it has been derived from its axioms. (p. 238)

Copi begins with a demonstration of a derived rule of inference for R.S. which may be used to validate infinitely many arguments within it, *i.e.*

DR 1 $P \supset Q, Q \supset R \ \vdash \ \sim(\sim RP)$

The demonstration of this derived rule requires just five *wffs*, the third of which is repeated, once in its unabbreviated form and once in its abbreviated form, thus:

S_1: $(P \supset Q) \supset [\sim(Q\sim R) \supset \ \sim(\sim RP)]$

S_2: $P \supset Q$

S_3: $\sim(Q\sim R) \supset \sim(\sim RP)$

S'_3: $(Q \supset R) \supset \sim(\sim RP)$

S_4: $Q \supset R$

S_5: $\sim(\sim RP)$

We can check that this series of S's is a demonstration as follows: S_1 is simply Axiom 3 of R.S. written in expanded form, thus

Ax. 3: $(P \supset Q) \supset [\sim(Q \bullet R) \supset \sim(R \bullet P)]$ and

S_1 : $(P \supset Q) \supset [\sim(Q\sim R) \supset \sim(\sim RP)]$

It is easy to see that S_1 has a '$\sim R$' wherever Ax. 3 has a 'R', and that both S_1 and Ax. 3 denote infinitely many *wffs* of R.S. The other lines in the sequence conform to what counts as a demonstration according to Copi's earlier definition of a demonstration. S_2 and S_4 are the premises of the argument, while S_3 follows from S_1 and S_2 by R1, while S_5 follows from S_3(or S'_3) and S_4 also by R1. When constructing proofs, we are used to writing the justification for each line alongside. The same annotation could be used here for convenience, although they form no part of the demonstration. (p. 238 - 239)

At this point Copi demonstrates, by way of an example how a derived rule of inference can be used in deriving theorems from axioms. According to the first theorem of R.S.

Th. 1 $\vdash \sim(\sim PP)$

This formula follows from Ax. 1 and Ax. 2 by means of DR 1, as follows:

$S_1: P \supset PP$ Ax. 1

$S_2: PP \supset P$ Ax. 2

$S_3: \sim(\sim PP)$ DR 1

That S_1 is identical to Ax. 1 is obvious. But S_2 is Ax. 2 because every *wff* of R.S. denoted by the syntactical formulation of Ax. 2 as '$PQ \supset P$' is an axiom and every *wff* of R.S. denoted by the syntactical expression '$PP \supset P$' is also denoted by '$PQ \supset P$'. On the other hand, S_3 really does follow from S_1 and S_2 by DR 1 because the latter validates any form of the argument

$P \supset Q$

$Q \supset R$

$\sim(\sim PP)$

Although the above sequence (S_1 - S_3) may be regarded as a 'proof' of Th. 1, it is not strictly speaking a demonstration because, recall, a demonstration involves the use of R1 exclusively. However we can insert the demonstration of DR 1 at S_3 into the sequence to produce a demonstration proper, thus:

$S_1: P \supset PP$	Ax. 1
$S_2: PP \supset P$	Ax. 2
$S_3: (P \supset PP) \supset [\sim(PP\sim P) \supset \sim(\sim PP)]$	Ax. 3
$S_4: \sim(PP\sim P) \supset \sim(\sim PP)$	R1
$S'_4: (PP \supset P) \supset \sim(\sim PP)$	
$S_5: \sim(\sim PP)$	R1

As it stands, this sequence is a demonstration; however as Copi points out, this demonstration results from our earlier 'proof' by certain changes indicated in the proof. Therefore, "[a] proof may be regarded as a prescription or recipe for the construction of a demonstration". (p. 239 - 240)

In an analogous fashion, further theorems of a deductive system may be derived from earlier theorems and not necessarily directly from their axioms. Again Copi shows this by means of an example. Consider a second theorem,

Th. 2 $\vdash \sim\sim P \supset P$

which follows directly from Th. 1 by definition, according to the following proof:

$S_1: \sim(\sim\sim P\sim P)$ Th. 1

$S'_1: \sim\sim P \supset P$ df.

This sequence is a proof which tells us how to construct a demonstration, if we so desire. Recall that the general statement of Th. 1 is

$\vdash \sim(\sim PP)$

which denotes every *wff* of this form, irrespective of which *wff* the syntactical variable 'P' denotes. Since $\sim(\sim\sim P\sim P)$ is of this form, it is one among the infinitely many provable formulae of R.S. that are called Th. 1. The demonstration of Th. 2 can be reconstructed as follows:

$S_1: \sim P \supset \sim P\sim P$	Ax. 1
$S_2: \sim P\sim P \supset \sim P$	Ax. 2
$S_3: (\sim P \supset \sim P\sim P) \supset [\sim(\sim P\sim P\sim\sim P) \supset \sim(\sim\sim P\sim P)]$	Ax. 3
$S_4: \sim(\sim P\sim P\sim\sim P) \supset \sim(\sim\sim P\sim P)$	R1
$S'_4: (\sim P\sim P \supset \sim P) \supset \sim(\sim\sim P\sim P)$	df.
$S_5: \sim(\sim\sim P\sim P)$	R1
$S'_5\ \sim\sim P \supset P$	df.

As can be seen, writing out demonstrations can be cumbersome, when proofs are shorter and easier to write down. In the remainder of the chapter therefore Copi provides proofs (rather than demonstrations) of further theorems. (p. 240 - 241)

Before doing so we might take conscience of the fact that Th. 2 can equally well be expressed as: $\vdash \sim P \text{ v } P$, which is one part of the principle or **law of the excluded middle**, which states that either a proposition is true or its negation is true. This should not be confused with the **principle of bivalence** which states that every proposition is either true or false. According to our definition of the symbol 'v', '$\sim P \text{ v } P$' is an abbreviation of '$\sim(\sim\sim P\sim P)$' of which Th. 2's '$\sim\sim P \supset P$' is an alternative abbreviation. In this form it represents one part of the principle or law of double negation. (p. 241)

The following theorems of R.S., together with their proofs, immediately below, are stated as follows:

Th. 3 $\vdash \sim(QR) \supset (R \supset \sim Q)$

$\vdash \sim\sim Q \supset Q$	Th. 2
$\vdash (\sim\sim Q \supset Q) \supset [\sim(QR) \supset \sim(R\sim\sim Q)]$	Ax. 3
$\vdash \sim(QR) \supset \sim(R\sim\sim Q)$	R1
$\vdash \sim(QR) \supset (R \supset \sim Q)$	df.

Th. 4 $\vdash R \supset \sim\sim R$

$\vdash \sim(\sim RR) \supset (R \supset \sim\sim R)$	Th. 3
$\vdash \sim(\sim RR)$	Th. 1
$\vdash R \supset \sim\sim R$	R1

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Th. 5 $\vdash (Q \supset P) \supset ({\sim}P \supset {\sim}Q)$

$\vdash\ {\sim}(Q{\sim}P) \supset ({\sim}P \supset {\sim}Q)$ Th. 3

$\vdash (Q \supset P) \supset ({\sim}P \supset {\sim}Q)$ df.

As Copi points out, Th. 5 is the principle of Transposition, meanwhile Th. 2 and T. 4. are different parts of the law of Double Negation. Both transportation and double negation are listed as rules of replacement in Critical Reasoning 07. Although both parts of the law of double negation, $P \supset {\sim}{\sim}P$ and ${\sim}{\sim}P \supset P$, have been proved as theorems, the law itself listed as, $P \equiv {\sim}{\sim}P$, has not yet been proven. The latter abbreviates the conjunction of both parts, $(P \supset {\sim}{\sim}P) \bullet ({\sim}{\sim}P \supset P)$, which would follow from Th. 4 and Th. 2 respectively; however the principle of Conjunction has yet to be established as a valid principle of inference in R.S. (p. 242 - 243)

Copi makes a few remarks that elucidate the meaning of the symbol '$\vdash$' when used with conjunction. '$\vdash P$' asserts that there is a sequence of *wffs* terminating in P that comprise of a demonstration. '$\vdash P$ and $\vdash Q$' however asserts that there are *two* sequences of *wffs*, one terminating in P and one terminating in Q, that comprise of two separate demonstrations. '$\vdash P \bullet Q$' on the other hand asserts that there is *one* sequence of *wffs* terminating in $P \bullet Q$. This latter assertion follows from the previous ones by the principle of Conjunction; however Copi postpones it for now, establishing it as DR 14, later. (p. 242)

The next derived rule, followed immediately by its proof is as follows:

DR 2 ${\sim}P \supset {\sim}Q \vdash Q \supset P$

$({\sim}P \supset {\sim}Q) \supset [{\sim}({\sim}Q) \supset {\sim}(Q{\sim}P)]$ Ax. 3

${\sim}P \supset {\sim}Q$ Premise

${\sim}({\sim}Q) \supset {\sim}(Q{\sim}P)$ R1

${\sim}({\sim}QQ)$ Th. 1

${\sim}(Q{\sim}P)$ R1

$Q \supset P$ df.

Copi points out that, although Th. 5, $\vdash (Q \supset P) \supset ({\sim}P \supset {\sim}Q)$ is one part of the principle of transportation, DR 2, ${\sim}P \supset {\sim}Q \vdash Q \supset P$ is not. Recall from Critical Reasoning 07 that transportation is listed as a rule of replacement as $(P \supset Q) \equiv ({\sim}Q \supset {\sim}P)$, which is the same as $(Q \supset P) \equiv ({\sim}P \supset {\sim}Q)$, which in turn is an abbreviation for $[(Q \supset P) \supset ({\sim}P \supset {\sim}Q)] \bullet [({\sim}P \supset {\sim}Q) \supset (Q \supset P)]$. The left hand part of this conjunct is Th. 5, but the right hand part is not DR 2. Clearly there an important difference between:

${\sim}P \supset {\sim}Q \vdash Q \supset P$ and $\vdash (P \supset {\sim}Q) \supset (Q \supset P)$

The first one above asserts that there is some sequence of *wffs*, "each of which is $\sim P \supset \sim Q$ or an axiom or follows from the preceding *wffs* by R1 and whose last *wff* is $Q \supset P$." The second one above however asserts that there is a sequence of *wffs*, "each of which is an axiom or follows from two preceding *wffs* by R1 and whose last *wff* is $\vdash (P \supset \sim Q) \supset (Q \supset P)$", which has not yet been established. (p. 242)

Nevertheless there is a connection between statements of the form $P \vdash Q$ and $\vdash P \supset Q$. Given the latter, the former may be deduced. According to Copi, the demonstration of $\vdash P \supset Q$ is sequence of *wffs* $S_1; S_2; \ldots ; S_k$ where S_k is $P \supset Q$. If we add P as S_{k+1} we can derive Q as S_{k+2} from S_k and S_{k+1} by R1. So although $\vdash P \supset Q$ follows from $P \vdash Q$, its proof is less simple, therefore Copi simply names it as

Metatheorem III. (the '*Deduction Theorem*')

and postpones its proof. Note that until it has been proved Metatheorem III cannot be assumed as part of R.S. (p. 242 - 243)

Three further derived rules, each followed immediately by their proof, are as follows:

DR 3 $P \supset Q \vdash RP \supset QR$

$(P \supset Q) \supset [\sim(QR) \supset \sim(RP)]$	Ax. 3
$P \supset Q$	Premise
$\sim(QR) \supset \sim(RP)$	R1
$RP \supset QR$	DR 2

DR 4 $P \supset Q, R \supset S \vdash \sim[\sim(QS)(PR)]$

$P \supset Q$	Premise
$SP \supset QS$	DR 3
$R \supset S$	Premise
$PR \supset SP$	DR 3
$\sim[\sim(QS)(PR)]$	DR 1

DR 5 $P \supset Q, Q \supset R, R \supset S \vdash P \supset S$

$R \supset S$	Premise
$(R \supset S) \supset (\sim S \supset \sim R)$	Th. 5
$\sim S \supset \sim R$	R1
$(\sim S \supset \sim R) \supset [\sim(\sim RP) \supset \sim(P \sim S)]$	Ax. 3

$\sim(\sim RP) \supset \sim(P\sim S)$	R1	
$P \supset Q$	Premise	
$Q \supset R$	Premise	
$\sim(\sim RP)$	DR 1	
$\sim(P\sim S)$	R1	
$P \supset S$	df.	

The form of DR 5 should be familiar to the reader by now as what Copi calls a "generalized" Hypothetical Syllogism, which has to be proved first before proving the more familiar Hypothetical Syllogism: $P \supset Q, Q \supset R \vdash P \supset R$ as DR 6. Before doing so Copi lists three further theorems, which he leaves to the reader to prove as an exercise. Jon Ross provided all the solutions that Copi did not, however they are no longer available at the original location.

Th. 6 $\vdash (R\sim\sim P) \supset (PR)$

Th. 7 $\vdash P \supset P$

Th. 8 $\vdash RP \supset PR$

It follows as a corollary to Th. 7 that

Th. 7, Cor. $\vdash P \text{ v} \sim P$

for which Copi provides the following proof:

$\sim P \supset \sim P$	Th. 7	
$\sim(\sim P\sim\sim P)$	df.	
$P \text{ v} \sim P$	df.	(p. 243 - 244)

DR 5 and Th. 7 may now be used together to prove the familiar Hypothetic Syllogism as

DR 6 $P \supset Q, Q \supset R \vdash P \supset R$

The following theorems and derived rules are required to prove Metatheorem III.

Th. 9 $\vdash \sim(PR) \supset \sim(RP)$

DR 7 $P \supset Q, R \supset S \vdash PR \supset QS$

Copi records two corollaries to DR 7 as:

DR 7, Cor. 1 $P \supset Q \vdash PR \supset QR$

DR 7, Cor. 2 $R \supset S \vdash PR \supset PS$

followed by

DR 8 $P \supset Q, P \supset R \vdash P \supset QR$

Theorem 10 below can be seen as one half of the principle of Association for '•' and its corollary as the other half:

Th. 10 $\vdash (PQ)R \supset P(QR)$

Th. 10, Cor. $\vdash P(QR) \supset (PQ)R$

Two further derived rules and the next theorem are listed as:

DR 9 $P \supset R, Q \supset S \vdash (P \text{ v } Q) \supset (R \text{ v } S)$

DR 10 $P \supset R, Q \supset R \vdash (P \text{ v } Q) \supset R$

Th. 11 $\vdash (P \text{ v } Q) \supset (Q \text{ v } P)$

Theorem 12 below can be seen as one half of the principle of Association for 'v' and its corollary as the other half:

Th. 12 $\vdash (P \text{ v } Q) \text{ v } R \supset P \text{ v } (Q \text{ v } R)$

Th. 12, Cor. $\vdash P \text{ v } (Q \text{ v } R) \supset (P \text{ v } Q) \text{ v } R$

Theorems 13 and 14 below are the two halves of the principle of Exportation; however according to Copi, before the principle itself can be derived from them, the principle of Conjunction must first be established (as DR 14).

Th. 13 $\vdash [P \supset (Q \supset R)] \supset [PQ \supset R]$

Th. 14 $\vdash [PQ \supset R] \supset [P \supset (Q \supset R)]$

Finally, one more derived rule and two theorems are required before proving the Deduction Theorem for R.S.

DR 11 $P \supset Q, P \supset (Q \supset R) \vdash P \supset R$

Th. 15 $\vdash P \supset (Q \supset PQ)$

Th. 16 $\vdash P \supset (Q \supset P)$ (p. 244 - 255)

Metatheorem III. *If* $P_1; P_2; \ldots; P_{n-1}; P_n \vdash Q$ *then* $P_1; P_2; \ldots; P_{n-1} \vdash P_n \supset Q$

Proof: (lightly edited for ease of reading) If we assume that $P_1; P_2; \ldots; P_{n-1}; P_n \vdash Q$, then we are assuming that there exists a demonstration or sequence of *wffs* $S_1; S_2; \ldots; S_s$ such that each S_i is either:

an axiom, or

a premise $P_i (i = 1; 2; \ldots; n)$ or follows from the previous two S's by R1

and that S_s is Q.

Consider the sequence of *wffs* $P_n \supset S_1$; $P_n \supset S_2$; … ; $P_n \supset S_s$. If we can 'fill in' the sequence of *wffs* before each $P_n \supset S_i$ in such a way that the total sequence is a demonstration from P_1; P_2; … ; P_{n-1} so that each line of the total sequence is either:

an axiom, or

a premise P_i(i = 1; 2; … ; n-1) or

follows from the two previous lines by R1,

then since $P_n \supset S_s$ is $P_n \supset Q$, we shall have a demonstration that P_1; P_2; … ; $P_{n-1} \vdash P_n \supset Q$. To prove that we can 'fill in' the sequence of *wffs* before each $P_n \supset S_i$ so as to get the desired demonstration. We then use the method of weak induction on the number of formulae $P_n \supset S_i$ involved.

α-case: In the case that i = 1 there is only the formula $P_n \supset S_1$ to consider, therefore by assumption S_1 is either an axiom or a premise P_i(i = 1; 2; … ; n)

- In the case that S_1 is an axiom we can fill in with the demonstration of Th. 16 $\vdash S_1 \supset (P_n \supset S_1)$ and S_1 itself. From these last two formulae we derive $P_n \supset S_1$ by R1. Therefore the total sequence of *wffs* up to and including $P_n \supset S_1$ is a demonstration that $\vdash P_n \supset S_1$ and hence that P_1; P_2; … ; $P_{n-1} \vdash P_n \supset S_1$.

- In the case that S_1 is a premise P_i(i = 1; 2; … ; n-1) we can fill in with the demonstration of Th. 16 $\vdash S_1 \supset (P_n \supset S_1)$ and S_1 itself. From these last two formulae we derive $P_n \supset S_1$ by R1. Therefore the total sequence of *wffs* up to and including $P_n \supset S_1$ is a demonstration that $S_1 \vdash P_n \supset S_1$. Since S_1 is a premise P_i(i = 1; 2; … ; n-1) we have a demonstration that P_1; P_2; … ; $P_{n-1} \vdash P_n \supset S_1$.

- In the case that S_1 is the premise P_n we can fill in with the demonstration of Th. 7 $\vdash P_n \supset P_n$ which is $\vdash P_n \supset S_1$. Therefore the total sequence of *wffs* up to and including $P_n \supset S_1$ is a demonstration that $\vdash P_n \supset S_1$ and hence that P_1; P_2; … ; $P_{n-1} \vdash P_n \supset S_1$.

β-case: Assume that we have correctly filled in all lines up to and including $P_n \supset S_{k-1}$, therefore we have a sequence of *wffs* which is a demonstration that P_1; P_2; … ; $P_{n-1} \vdash P_n \supset S_{k-1}$. Under this assumption we can show how to fill in the sequence of *wffs* up to and including $P_n \supset S_k$ which will then be a demonstration that P_1; P_2; … ; $P_{n-1} \vdash P_n \supset S_k$. By assumption S_k is either an axiom or a premise P_i(i = 1; 2; … ; n) or the result of the application of R1 to the two previous S's in the original demonstration, say S_i and S_j such that $i, j < k$.

- In the case that S_k is an axiom, we can insert the demonstration of Th. 16 $\vdash S_k \supset (P_n \supset S_k)$ and S_k itself and derive $P_n \supset S_k$ by R1. The completed sequence will then be a demonstration that $P_1; P_2; \ldots ; P_{n-1} \vdash P_n \supset S_k$.

- In the case that S_k is a premise $P_i(i$ = 1; 2; … ; n-1), we can insert the demonstration of Th. 16 $\vdash S_k \supset (P_n \supset S_k)$ and S_k itself and derive $P_n \supset S_k$ by R1. The completed sequence will then be a demonstration that $P_1; P_2; \ldots ; P_{n-1} \vdash P_n \supset S_k$.

- In the case that S_k is P_n, we can insert the demonstration of Th. 7 $\vdash P_n \supset P_n$ which is $\vdash P_n \supset S_k$. The completed sequence will then be a demonstration that $P_1; P_2; \ldots ; P_{n-1} \vdash P_n \supset S_k$.

- In the case that S_k resulted from the application of R1 to the two previous S's in the original demonstration, say S_i and S_j such that $i, j < k$ and S_i is of the form $S_j \supset S_k$, then since by the β-case assumption we have correctly filled in all the lines up to and including $P_n \supset S_j$ and $P_n \supset (S_j \supset S_k)$, therefore by DR 11 $P \supset Q, P \supset (Q \supset R) \vdash P \supset R$ we have:

  $$P_n \supset S_j, P_n \supset (S_j \supset S_k) \vdash P_n \supset S_k$$

  If we insert the demonstration of this derived rule whose last line is $P_n \supset S_k$, then the entire sequence will be a demonstration that $P_1; P_2; \ldots ; P_{n-1} \vdash P_n \supset S_k$.

Now by weak induction we may conclude that we can fill in *any* number of lines $P_n \supset S_i$ in such a way that the resulting sequence will be a demonstration that $P_1; P_2; \ldots ; P_{n-1} \vdash P_n \supset S_i$. We can then do the same for the demonstration that $P_1; P_2; \ldots ; P_{n-1};\ P_n \vdash Q$, no matter how many lines S_1 to S_s are involved. And since S_s is Q we can construct a demonstration that $P_1; P_2; \ldots ; P_{n-1};\ P_n \vdash Q$. (p. 245 - 246)

Metatheorem III. Corollary *If* $P \vdash Q$ *then* $\vdash P \supset Q$

Metatheorem III. a.k.a. the Deduction Theorem is true in general for *any* propositional calculus that has only the rule *Modus Ponens* and demonstrations for $P \supset P$, $P \supset (Q \supset P)$ and $(P \supset Q) \supset \{[P \supset (Q \supset R)] \supset (P \supset R)\}$.

The Deduction Theorem ('D.T.') may be used in the proof of DR 6 as follows. First we prove the relatively trivial DR 6' $P \supset Q$, $Q \supset R$, $P \vdash R$ thus:

$P \supset Q$	Premise
P	Premise
Q	R1
$Q \supset R$	Premise
R	R1

then from DR 6' we apply D.T. once so that the two line poof reads

$P \supset Q, Q \supset R, P \vdash R$ DR 6'

$P \supset Q, Q \supset R \vdash P \supset R$ D.T.

Clearly this is a much more economical way of proving DR 6 on the basis of the 'old' proof for DR 6'; however recall that we had to establish the more difficult proofs for our earlier theorems in order to proof the Deduction Theorem in the first place. (p. 246 -247)

Here Copi lists some further theorems and derived rules of R.S.

Th. 17 $\vdash P \supset (Q \text{ v } P)$

Th. 17, Cor. $\vdash P \supset (P \text{ v } Q)$

Th. 18 $\vdash (P \text{ v } Q)R \supset (PR \text{ v } QR)$

Th. 18 is one part of the principle of Distribution of '•' with respect to 'v'. Three derived rules follow:

DR 12 $P \supset {\sim}Q \vdash P \supset {\sim}(QR)$

DR 13 $P \supset {\sim}R \vdash P \supset {\sim}(QR)$

DR 14 $P, Q \vdash PQ$

DR 14 is the principle of Conjunction that was postponed earlier. Using DR 14 we can derive principle of Double Negation from Th. 2 $\vdash {\sim\sim}P \supset P$ and Th. 4 $\vdash R \supset {\sim\sim}R$, thus:

Th. 19 $\vdash P \equiv {\sim\sim}P$

Although Th. 19 allows us to replace ${\sim\sim}P$ by P on a line by itself, it does not allow such replacement within a larger *wff*. Although the inference from the *wff* $\vdash (\ldots\ {\sim\sim}P \ldots)$ to $\vdash (\ldots\ P \ldots)$ is valid we need to prove it so within R.S. before we can use it. The legitimacy of such a move is asserted by Metatheorem IV., however before stating and proving it we need to establish the following three derived rules:

DR 15 $P \equiv Q \vdash {\sim}P \equiv {\sim}Q$

DR 16 $P \equiv Q, R \equiv S \vdash PR \equiv QS$

DR 16, Cor. $P \equiv Q, R \equiv S \vdash P \text{ v } R \equiv Q \text{ v } S$

Copi leaves it as an exercise to prove these. Jon Ross has provided formal proofs for them however they are no longer in their original location. (p. 247 - 248)

Metatheorem IV. (The Rule of Replacement) *If* P_1; P_2; … ; P_n *are any wffs* and Q *is another wff that does not occur in any* P_i *and if* S *is a wff that contains no other components other than* Q *and* P_i $(1 \leq i \leq n)$, *then where* S^* *is a wff that that results from replacing any number of occurrences of* Q *in* S *by* R, *then* $Q \equiv R \vdash S \equiv S^*$.

Proof: (lightly edited) Copi uses the method of strong induction on the number of symbols in S, counting each occurrence of •, ~, Q or any P_i as a single symbol.

α-case: For n = 1, S is either Q alone or P_i alone.

- In the case that S is Q and S^* is R, it is obvious that $Q \equiv R \vdash Q \equiv R$ and that $Q \equiv R \vdash S \equiv S^*$.

- In the case that S is Q and S^* is also Q, then $\vdash Q \equiv Q$ by Th. 7 and DR 14 and $\vdash S \equiv S^*$, hence $Q \equiv R \vdash S \equiv S^*$.

- In the case that S is one of P_i, then S^* is also the same P_i. Since $\vdash P_i \equiv P_i$ by Th. 7 and DR 14 and $\vdash S \equiv S^*$, hence $Q \equiv R \vdash S \equiv S^*$.

β-case: Assume Metatheorem IV. true for any S containing $< n$ symbols, then for any S containing exactly n symbols (n >1), S must be either $\sim S_1$ or $S_1 \bullet S_2$.

- In the case that S is $\sim S_1$, because S contains n symbols, S_1 will contain $< n$ symbols, therefore by the β-case assumption $Q \equiv R \vdash S_1 \equiv S_1^*$, where S_1^* is a *wff* that results from replacing any number of occurrences of Q in S_1 by R. But since $S_1 \equiv S_1^* \vdash \sim(S_1) \equiv \sim(S_1^*)$ by DR 15 and because any $\sim(S_1^*)$ is the same as a $(\sim S_1)^*$ or just S^*, it follows that $Q \equiv R \vdash S \equiv S^*$.

- In the case that S is $S_1 \bullet S_2$, each of S_1 and S_2 contains $< n$ symbols, therefore by the β-case assumption $Q \equiv R \vdash S_1 \equiv S_1^*$ and $Q \equiv R \vdash S_2 \equiv S_2^*$. However by DR 16 $S_1 \equiv S_1^*, S_2 \equiv S_2^* \vdash S_1 \bullet S_2 \equiv S_1^* \bullet S_2^*$, and since any S^* is an $S_1^* \bullet S_2^*$, it follows that $Q \equiv R \vdash S \equiv S^*$.

Therefore by the method of strong induction we may infer that, no matter the number of symbols in S, $Q \equiv R \vdash S \equiv S^*$. (p. 248 - 249)

Metatheorem IV. Corollary *If* Q, R, S *and* S^* *are as they are in Metatheorem* IV. *then*

$$Q \equiv R, S \vdash S^*$$

According to Copi, the poof of this is obvious and it is. What is remarkable is how the remaining Elementary Rules of Inference and Logically Equivalent Rules of Replacement introduced in Critical Reasoning 07 almost "fallout" of R.S. at this point, given its further development. Only *Modus Ponens* was assumed as the primitive rule, R1. The Hypothetical Syllogism has already been established as DR 6 and the principle of Conjunction as DR 14. The remaining six Elementary Rules of Inference are easily proved as derived rules of R.S. Copi lists them together as follows:

DR 17 $P \supset Q, \sim Q \vdash \sim P$	*Modus Tollens*
DR 18 $P \vee Q, \sim P \vdash Q$	Disjunctive Syllogism
DR 19 $PQ \vdash P$	Simplification
DR 19, Cor. $PQ \vdash Q$	

DR 20 $(P \supset Q)(R \supset S), P \vee R \vdash Q \vee S$ Constructive Dilemma

DR 21 $(P \supset Q)(R \supset S), \sim Q \vee \sim S \vdash \sim P \vee \sim R$ Destructive Dilemma

DR22 $P \vdash P \vee Q$ Addition

The ten Rules of Replacement which were all stated as equivalences are easily established. Double Negation has already been proved as Th. 19. Those that involve Commutation or Association can be obtained by applying DR 14 to theorems and corollaries that have already been established.

Th. 20 $\vdash P \vee Q \equiv Q \vee P$ Commutation of 'v'

Th. 21 $\vdash PQ \equiv QP$ Commutation of '•'

From Th. 21 it follows that $\vdash (P \equiv Q) \equiv (Q \equiv P)$, which according to Copi can be used "without special mention" to prove:

Th. 22 $\vdash [P \vee (Q \vee R)] \equiv [(P \vee Q) \vee R]$ Association of 'v'

Th. 23 $\vdash P(QR) \equiv (PQ)R$ Association of '•' (p. 249)

According to Copi, "The principle of Transposition is obtainable by DR 14 from Theorem 5 and the result of applying the Deduction Theorem to DR. 2."

Th. 24 $\vdash (P \supset Q) \equiv (\sim Q \supset \sim P)$ Transposition

Jon Ross was helpful in spelling out just what Copi had in mind, thus:

1. $(P \supset Q) \supset (\sim Q \supset \sim P)$ Th. 5
2. $\sim Q \supset \sim P \vdash P \supset Q$ DR 2
3. $(\sim Q \supset \sim P) \supset (P \supset Q)$ 2 D.T.
4. $[(P \supset Q) \supset (\sim Q \supset \sim P)][(\sim Q \supset \sim P) \supset (P \supset Q)]$ 1, 3 DR 14
5. $(P \supset Q) \equiv (\sim Q \supset \sim P)$ df.

Fortunately, the proof of the principle of Exportation is a little easier.

Th. 25 $\vdash [(PQ) \supset R] \equiv [P \supset (Q \supset R)]$ Exportation

Copi simply lists the remaining theorems of R.S. and leaves it as an exercise for the reader to prove.

Th. 26 $\vdash P \equiv PP$ Tautology

Th. 26, Cor. $\vdash P \equiv P \vee P$ Tautology

Th. 27 $\vdash \sim(PQ) \equiv (\sim P \vee \sim Q)$ De Morgan's Theorem

Th. 28 $\vdash \sim(P \vee Q) \equiv (\sim P \sim Q)$ De Morgan's Theorem

Th. 29 $\vdash (P \supset Q) \equiv (\sim P \vee Q)$ Material Implication

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Th. 30 $\vdash P(Q \text{ v } R) \equiv PQ \text{ v } PR$ Distribution of '•' over 'v'

Th. 30, Cor. $\vdash (P \text{ v } Q)R \equiv PR \text{ v } QR$ Distribution of '•' over 'v'

Th. 31 $\vdash (P \equiv Q) \equiv (PQ \text{ v } {\sim}P{\sim}Q)$ Material Equivalence

Th. 32 $\vdash P \text{ v } QR \equiv (P \text{ v } Q)(P \text{ v } R)$ Distribution of 'v' over '•'

With all of these theorems established, R.S. has been shown to contain all the logical principles relied upon in The Method of Deduction (Critical Reasoning 07). Furthermore, because it contains the Deduction Theorem as well as the principle of Double Negation it is also adequate for the methods of Condition Proof and Indirect Proof (Critical Reasoning 09). All that remains to be proved in the last section is that R.S. is *deductively complete*. (p. 250)

Deductive Completeness

In Theorems 22, 23 and 30 we have already established the Association of 'v' and '•' and the Distribution of '•' with respect to 'v'; however as stated, these properties involve exactly three *wffs*. In proving the deductive completeness of R.S. in general it is desirable to establish more general versions of the Association and Distribution principles. These Copi sets out as the following three Metatheorems; the first of which establishes the general Association and Commutation principles of Conjunction ('•'). Informally the latter states that, "no matter in what order or grouping any number of *wffs* are conjoined, the resulting *wff* will be equivalent to the result of conjoining them in any other order or groping". (p. 250 - 251)

Formally stated then:

Metatheorem V. *Let $P_1; P_2; \ldots ; P_n$ be any wffs and let Q and R be any wffs constructed out of them by means of* '•'. *If each P_i $(1 \leq i \leq n)$ occurs exactly once in each of the wffs Q and R, then*

$$\vdash Q \equiv R.$$

Proof: (lightly edited) Here Copi uses the method of strong induction on the number conjuncts P_i in Q and R.

α-case: For n = 1, Q and R are the same *wff* P_1, therefore $\vdash Q \equiv R$ by Th. 7 and DR 14.

β-case: Assume Metatheorem V. true for every $k < n$ conjuncts $P_1; P_2; \ldots ; P_k$. Now consider Q and R, each constructed out of $n > 1$ conjuncts $P_1; P_2; \ldots ; P_n$. Say Q is $S \bullet T$ and R is $X \bullet Y$.

Each of the *wffs* S and T contains at least one of the conjuncts $P_1; P_2; \ldots ; P_n$. We can assume that P_1 is a conjunct of S because, if not, we could simply apply commutation of '•' (Th. 21) and relabel the expression to obtain $\vdash Q \equiv S \bullet T$ so that S does contain P_1 as a conjunct.

Because T contains at least one of $P_2; P_3; \ldots ; P_n$ as a conjunct, S must contain $< n$ of the conjuncts $P_1; P_2; \ldots ; P_n$. Hence, either S is P_1 and $\vdash Q \equiv P_1 \bullet T$ or, by the β-case assumption, $\vdash S \equiv P_1 \bullet S'$ where S' is a *wff* that contains all of the conjuncts of S except P_1. In the latter case then, by the corollary to Metatheorem IV, we have

$$\vdash Q \equiv (P_1 \bullet S') \bullet T$$

By Association of '•' (Th. 23) and the corollary to Metatheorem IV, we have it that

$$\vdash Q \equiv P_1 \bullet (S' \bullet T)$$

In either case there must be a *wff*, call it T', such that

$$\vdash Q \equiv P_1 \bullet T'$$

And by the same reasoning, there must be a *wff*, call it Y', such that

$$\vdash R \equiv P_1 \bullet Y'$$

Both T' and Y' contain n - 1 conjuncts (P_2; P_3; … ; P_n), therefore by the β-case assumption

$$\vdash T' \equiv Y'$$

By tautology (Th. 7, $\vdash P \supset P$) and Conjunction (DR 14), we have

$$\vdash P_1 \equiv P_1$$

from which it follows by DR 16 that

$$\vdash P_1 \bullet T' \equiv P_1 \bullet Y'$$

and by the corollary to Metatheorem IV

$$\vdash Q \equiv R$$

Therefore Metatheorem V follows by strong induction. (p. 251 - 252)

The next principle that Copi addresses is the general Association and Commutation of Disjunction, 'v'. No matter what the order or grouping of *wffs* connected by 'v', the resulting disjunction or '**logical sum**' will be equivalent to that of connecting them, by 'v' in any other order or grouping. Formally stated then:

Metatheorem VI. *Let* P_1; P_2; … ; P_n *be any wffs and let* Q *and* R *be any wffs constructed out of them by means of* 'v'. *If each* P_i $(1 \leq i \leq n)$ *occurs exactly once in each of the wffs* Q *and* R, *then* $\vdash Q \equiv R$.

Copi leaves the proof of this as an exercise for the reader; however Jon Ross constructed a proof along similar lines to the one above which relies on strong induction on the number of disjucnts in P_i, Q and R. Unfortunately, Ross' proof can no longer be found in the original location.

In the last of this troika of metatheorems we wish to establish a generalised statement of the Distribution of conjunction '•' with respect to disjunction 'v'. Formally then:

Metatheorem VII. *If* Q *is the logical sum of* P_1; P_2; … ; P_n (*i.e.* $P_1 \vee P_2 \vee \ldots \vee P_n$) *with association to the left as a given convention, and* S *being the logical sum of* P_1R; P_2R; … ; P_nR, *then* $\vdash QR \equiv S$.

Proof: (lightly edited) This time Copi uses the method of weak induction on the number of disjucnts, P_1; P_2; … ; P_n.

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α-case: For n = 1, Q is P_1 therefore QR is P_1R and S is also P_1R. By tautology (Th. 7, $\vdash P \supset P$) and Conjunction (DR 14), $\vdash P_1R \equiv P_1R$ which is the same as $\vdash QR \equiv S$.

β-case: Assume Metatheorem VII. true for k disjucnts $P_1; P_2; \ldots ; P_k$. Now let Q be the logical sum or disjunction of $P_1; P_2; \ldots ; P_k;\ P_{k+1}$ and let S be the logical sum of $P_1R; P_2R; \ldots ; P_kR;\ P_{k+1}R$. According to the following argument:

$\vdash (P_1 \text{ v } P_2 \text{ v} \ldots \text{v } P_k)R \equiv P_1R \text{ v } P_2R \text{ v} \ldots \text{v } P_kR$ by the β-case assumption

$\vdash P_{k+1}R \equiv P_{k+1}R$ Th. 7 and DR 14

$\vdash (P_1 \text{ v } P_2 \text{ v} \ldots \text{v } P_k)R \text{ v } P_{k+1}R \equiv (P_1R \text{ v } P_{2R} \text{ v} \ldots \text{v } P_kR) \text{ v } P_{k+1}R$ DR 16, Cor.

$\vdash [(P_1 \text{ v } P_2 \text{ v} \ldots \text{v } P_k) \text{ v } P_{k+1}]R \equiv (P_1 \text{ v } P_2 \text{ v} \ldots \text{v } P_k)R \text{ v } P_{k+1}R$ Th. 30, Cor.

$\vdash [(P_1 \text{ v } P_2 \text{ v} \ldots \text{v } P_k) \text{ v } P_{k+1}]R \equiv (P_1R \text{ v } P_2R \text{ v} \ldots \text{v } P_kR) \text{ v } P_{k+1}R$ MT IV, Cor.

which by the convention of association to the left, can be written as

$\vdash (P_1 \text{ v } P_2 \text{ v} \ldots \text{v } P_k \text{v } P_{k+1})R \equiv P_1R \text{ v } P_2R \text{ v} \ldots \text{v } P_kR \text{ v } P_{k+1}R$

which is the same as $\vdash QR \equiv S$, where Q contains k+ 1 disjucnts.

Therefore Metatheorem VI therefore follows by weak induction. (p. 252 - 253)

According to Copi, "[t]o prove that R.S. is deductively complete, we show that all tautologies are demonstrable as theorems in the system. Since all tautologies are expressible as *wffs* of R.S. (by M.T. I), the deductive completeness of R.S. is expressed as: if S is a *wff* that is a tautology, then $\vdash S$". Fortunately we know of a method by which to decide whether a particular *wff* is a tautology. Recall that the truth table of a tautology has only T's in the last column. In general, any *wff* S has a corresponding truth table with as many initial columns as there are distinct propositional symbols in S. Were, for example, there are n of them, $P_1; P_2; \ldots ; P_n$, the corresponding truth table of the tautology S, will have general form:

P_1	P_2	$\cdots$	P_n	S
T	T	$\cdots$	T	T
T	T	$\cdots$	F	T
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
F	F	$\cdots$	T	T
F	F	$\cdots$	F	T

Note that the truth table has 2^n rows corresponding to the different assignments of T's and F's to each of the P_i's, "one row for every possible assignment". As expected for a tautology, only T's appear under column S meaning that every possible assignment of T's and F's to each of the P_i's must result in the assignment of a T under S. According to Copi, in order to show that each S results in $\vdash S$, we must establish the following:

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1. Each row of the corresponding truth table that represents an assignment of truth values to each of the P_i's can be represented by a *wff* of R.S., the first row by Q_1, the second by Q_2, …, the ith row by Q_i … and the last or 2^nth row by Q_{2^n};

2. if Q_1; Q_2; … ; Q_{2^n} are all of the 2^n *wffs* representing all the possible assignments of T's and F's to each of the P_i's, then $\vdash (Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^n})$; and

3. if a particular assignment of T's and F's to a P_i results in the assignment of a T to S, where Q_j represents that assignment, then $\vdash Q_j \supset S$. (p. 253)

Copi demonstrates that the above suffice to prove that $\vdash S$ as follows: Since our truth table has all T's under column S, it follows that $\vdash Q_1 \supset S$; $\vdash Q_2 \supset S$; … ; $\vdash Q_{2^n} \supset S$. By 2^n - 1 uses of DR 10 ($P \supset R, Q \supset R \vdash (P \vee Q) \supset R$) we have $\vdash (Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^n}) \supset$ S. Therefore once we establish that $\vdash (Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^n})$, we obtain $\vdash S$ by R1. Firstly though, we must show that each possible assignment of T's and F's to a P_i of the set $\{P_1; P_2; \ldots ; P_n\}$ can be represented by a *wff* of R.S. The concept of 'can be represented by a *wff*' however stands in need of a definition, which Copi supplies.

> **Defn.** A *wff* is said to *represent* a particular assignment of truth values to propositional symbols P_1; P_2; … ; P_n if and only if (on the normal interpretation of our operator symbols) that truth value assignment is the only one which makes that *wff* true. (p. 254)

If we assign a T to every P_i, then this assignment is represented by the conjunction $P_1 \bullet P_2 \bullet \ldots \bullet P_n$, which Copi denotes as 'Q_1'. If instead we assign a T to every P_i except to P_n to which we assign an F then then this assignment is represented by the conjunction $P_1 \bullet P_2 \bullet \ldots \bullet P_{n-1} \bullet \sim P_n$, which Copi denotes as 'Q_2'. We can continue in this fashion for every other possible assignment of truth values until we reach the last row of the table, represented by the conjunction $\sim P_1 \bullet \sim P_2 \bullet \ldots \bullet \sim P_n$, which, following Copi, would be denoted 'Q_{2^n}'. Therefore we have shown that any row of every truth table can be represented by a *wff* of R.S. This satisfies point 1. above. The second point is expressed as

Metatheorem VIII. *If Q_1; Q_2; … ; Q_{2^n} represent all possible distinct assignments of truth values to the n distinct propositional symbols P_1; P_2; … ; P_n, then* $\vdash (Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^n})$.

Proof: (lightly edited) Copi uses the method of weak induction on the number of P_i's involved.

α-case: For n = 1, 2^n = 2 which means that Q_1 is P_1 and Q_2 is $\sim P_1$, which exhausts all possible combinations, we have $\vdash P_1 \vee \sim P_1$, which by Th. 7, Cor. ($\vdash P \vee \sim P$) yields $\vdash (Q_1 \vee Q_2)$.

β-case: Assume Metatheorem VIII. true for P_1; P_2; … ; P_k. If we consider the set $\{P_1; P_2; \ldots ; P_k; P_{k+1}\}$ where Q_1; Q_2; … ; Q_{2^k} represent all the possible distinct assignments of truth values to P_1; P_2; … ; P_k, then we have $\vdash (Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^k})$ by the β-case assumption. According to the following argument:

$\vdash P_{k+1} \vee \sim P_{k+1}$ — Th. 7, Cor.

$\vdash (Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^k})(P_{k+1} \vee \sim P_{k+1})$ — DR 14

$\vdash \left(Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^k}\right)(P_{k+1} \vee \sim P_{k+1}) \equiv$
$\left(Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^k}\right)P_{k+1} \vee \left(Q_1 \vee Q_2 \ \vee \ldots \vee Q_{2^k}\right)\sim P_{k+1}$ — Th. 30

$\vdash (Q_1 \vee Q_2 \ \vee ... \vee Q_{2^k})P_{k+1} \vee (Q_1 \vee Q_2 \ \vee ... \vee Q_{2^k}){\sim}P_{k+1}$ MT IV, Cor.

$\vdash (Q_1 \vee Q_2 \ \vee ... \vee Q_{2^k})P_{k+1} \equiv (Q_1P_{k+1} \vee Q_2P_{k+1} \vee ... \vee Q_{2^k}P_{k+1})$ MT VII

$\vdash (Q_1 \vee Q_2 \ \vee ... \vee Q_{2^k}){\sim}P_{k+1} \equiv (Q_1{\sim}P_{k+1} \vee Q_2{\sim}P_{k+1} \vee ... \vee Q_{2^k}{\sim}P_{k+1})$ MT VII

$\vdash (Q_1P_{k+1} \vee Q_2P_{k+1} \vee ... \vee Q_{2^k}P_{k+1}) \vee (Q_1{\sim}P_{k+1} \vee Q_2{\sim}P_{k+1} \vee ... \vee Q_{2^k}{\sim}P_{k+1})$
MT IV, Cor.

$\vdash Q_1P_{k+1} \vee Q_1{\sim}P_{k+1} \vee Q_2P_{k+1} \vee Q_2{\sim}P_{k+1} \vee ... \vee Q_{2^k}P_{k+1} \vee Q_{2^k}{\sim}P_{k+1}$ MT VII, Cor.

If we count the number of disjucnts in the previous line we get 2^k + 2^k = $2 \cdot 2^k$ = 2^{k+1} distinct disjucnts, each of which represents a distinct assignment of truth values to each of the P_1; P_2; ... ; P_{k+1}. Each Q_iP_{k+1} and each $Q_i{\sim}P_{k+1}$ are alternate Q_i's of which there are 2^{k+1} Q_i's which represent all the possible assignments of truth values to P_1; P_2; ... ; P_{k+1}. Hence $\vdash (Q_1' \vee Q_2' \vee ... \vee Q_{2^{k+1}}')$. Metatheorem VIII therefore follows by weak induction. (p. 254 - 255)

Copi's next move is to prove that if the truth value assignment represented by Q_j assigns a T to the corresponding entry under the column S, then $\vdash Q_j \supset S$. This can be proved more generally by also including the case in which an F is assigned to S instead. This is stated and proved as Metatheorem IX below.

Metatheorem IX. *If Q_j represents any possible assignment of truth values to the n propositional symbols P_1; P_2; ... ; P_n, and if S is a wff that has no propositional components other than P_i $(1 \leq i \leq n)$, then if the truth value assignment represented by Q_j assigns a* T *to S, then $\vdash Q_j \supset S$. If however the truth value assignment represented by Q_j assigns a* F to *S, then $\vdash Q_j \supset {\sim}S$.*

Proof: (lightly edited) Copi uses the method of strong induction on the number of symbols in S such that each occurrence of •, ∼ and any P_i counts as a single symbol.

α-case: For n = 1, S must consist of a single symbol, and since it is a *wff* it must be a $P_i (1 \leq i \leq n)$.

- In the case that Q_j assigns a T to S, *i.e.* to P_i, that P_i rather ${\sim}P_i$ must be a disjucnt of Q_j. By MT V, $\vdash Q_j \equiv P_iR$, where R is the conjunction of all disjucnts of Q_j except P_i. Copi then argues that

  $\vdash P_iR \supset P_i$ Ax. 2

  $\vdash Q_j \supset P_i$ MT IV, Cor.

  which is the same as $\vdash Q_j \supset S$.

- In the case that Q_j assigns a F to S, *i.e.* to P_i, ${\sim}P_i$ must be a disjucnt of Q_j. By MT V, $\vdash Q_j \equiv {\sim}P_iR$, where R is the conjunction of all disjucnts of Q_j except ${\sim}P_i$. Copi then argues that

  $\vdash {\sim}P_iR \supset {\sim}P_i$ Ax. 2

  $\vdash Q_j \supset {\sim}P_i$ MT IV, Cor.

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which is the same as $\vdash Q_j \supset \sim S$.

β-case: Assume Metatheorem IX. true for any S containing $k < n$ number of symbols. For any $n > 1$ symbols the *wff* S is ether $S_1 \bullet S_2$ or $\sim S_3$.

- In the case that Q_j assigns a T to S, Q_j must assign a T to S_1 and to S_2. Since S_1 and S_2 contain fewer than n symbols each, Copi then argues that

$\vdash Q_j \supset S_1$ by the β-case assumption

$\vdash Q_j \supset S_2$ by the β-case assumption

$\vdash Q_j \supset S_1 \bullet S_2$ DR 8

which is the same as $\vdash Q_j \supset S$.

- In the case that Q_j assigns a F to S, Q_j must assign a F to S_1 or to S_2. If an F is assigned to S_1 then $\vdash Q_j \supset \sim S_1$ by the β-case assumption, hence by DR 12 $\vdash Q_j \supset \sim(S_1 \bullet S_2)$, which is the same as $\vdash Q_j \supset \sim S$. If, on the other hand, an F is assigned to S_2 then $\vdash Q_j \supset \sim S_2$ by the β-case assumption, and hence by DR 13 $\vdash Q_j \supset \sim(S_1 \bullet S_2)$, which is the same as $\vdash Q_j \supset \sim S$.

- In the case that S is $\sim S_3$:

  EITHER Q_j assigns a T to S so that Q_j in turn must assign a F to S_3, in which case by the β-case assumption $\vdash Q_j \supset \sim S_3$, which is the same as $\vdash Q_j \supset S$.

  OR Q_j assigns a F to S so that Q_j in turn must assign a T to S_3, in which case by the β-case assumption $\vdash Q_j \supset S_3$. However $\vdash S_3 \supset \sim\sim S_3$ by Th. 4, so that by DR 6 $\vdash Q_j \supset \sim\sim S_3$, which is the same as $\vdash Q_j \supset \sim S$.

Metatheorem IX therefore follows by strong induction. (p. 255 - 256)

Finally all the elements are in place to state and prove the deductive completeness of R.S. as follows:

Metatheorem X. *R.S. is deductively complete* (*i.e. if S is a tautology, then* $\vdash S$).

Proof: "If S is a tautology then every possible assignment of truth valves to its components P_1; P_2; … ; P_n must assign a T to S. Hence by MT IX:

$\vdash Q_1 \supset S$
$\vdash Q_2 \supset S$
…………….
$\vdash Q_{2^n} \supset S$

where Q_1; Q_2; … ; Q_{2^n} represent all possible distinct assignments of truth values to P_1; P_2; … ; P_n. Now by 2^n- 1 uses of DR 10, we have

$$\vdash (Q_1 \vee Q_2 \vee \ldots \vee Q_{2^n}) \supset S$$

and by MT VIII,

$$\vdash (Q_1 \vee Q_2 \vee \ldots \vee Q_{2^n})$$

From these we derive $\vdash S$ by R1, which completes the proof of Metatheorem X." (p. 256 - 257)

According to Copi, the **decision problem** for any deductive system is to state an effective criterion for deciding whether or not any statement or *wff* is a theorem of the system. We have already established by MT II that R.S. is analytic and by MT X that is deductively complete; the method of truth tables therefore constitutes a solution to the decision problem. Since, by MT II, only tautologies are theorems and since, by MT X, all tautologies are theorems, truth tables enable us to decide whether a *wff* is a theorem. Moreover the proofs up to and including MT X not only assure us of the existence of a demonstration for every tautologous *wff*, they actually provide a method for constructing each demonstration. Although constructing such a demonstration along the lines of the proof for deductive completeness may be longer than more inventive and elegant alternatives, it is an effective method that can be implemented by a purely mechanical procedure, in a finite number of steps and that without the need for inventiveness or elegance. (p. 257)

In our presentation of the Method of Deduction in Critical Reasoning 07 and Conditional Proofs in Critical Reasoning 09, which correspond to Copi's Ch. 3, we claimed that the validity of any argument could be proved using only the 9 Elementary Rules of Inference and the 10 Logically Equivalent Rules of Replacement, supplemented by the rules of Conditional and Indirect Proof. According to Copi, we are now in a position to substantiate that claim. This is equivalent to the assertion that the Method of Deduction is deductively complete. This can be shown by demonstrating that every argument that can be proved valid in R.S. can also be proved valid by the Method of Deduction (supplemented by the rules of Conditional and Indirect Proof). (p. 257)

For any argument $P_1; \ldots; P_n \therefore Q$ that can be proved valid in R.S. there exists a demonstration in R.S. for the derived rule $P_1; \ldots; P_n \vdash Q$. By n uses of the Deduction Theorem, Exportation and the Rule of Replacement, we have $\vdash P \supset Q$ where P is a conjunction of the premises $P_1; \ldots; P_n$. Since R.S. is analytic (*i.e.* if $\vdash P$ then P is a tautology) $P \supset Q$ is a tautology and $P \bullet {\sim}Q$ must be a contradiction. But there exists a formal proof for the argument

$$P_1; \ldots; P_n; {\sim}Q \therefore P \bullet {\sim}Q \qquad \ldots ①$$

using just the Method of Deduction. Hence there exists a formal proof, again using just the Method of Deduction, of

$$P_1; \ldots; P_n; {\sim}Q \therefore N \qquad \ldots ②$$

where N is a disjunctive form of $P \bullet {\sim}Q$. However because $P \bullet {\sim}Q$ is a contradiction, N must be a disjunction in which every disjucnt contains a contradiction as a conjunct. Hence by formally proving the validity of

$$q \vee [(p \bullet {\sim}p) \bullet r] \therefore q$$

the formal proof of validity for ② can be used to formally prove

$$P_1; \ldots; P_n; {\sim}Q \therefore N_1 \qquad \ldots ③$$

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where N_1 is a single disjucnt of N. If N_1 is not itself an explicit contradiction it must be a conjunction containing a contradiction as one of its conjuncts. Hence by the use of Commutation, Simplification (and Association, if needed) the formal proof of validity for ③ can be used to derive a contradiction from the premises $P_1; \dots ; P_n; \sim Q$. Such a derivation constitutes an Indirect Proof of the validity of the original argument $P_1; \dots ; P_n \therefore Q$. Hence any argument that can be proved valid in R.S. can also be proved valid by the Method of Deduction (supplemented by the rules of Conditional and Indirect Proof, if required). Therefore the latter is also a deductively complete system of logic. (p. 257 - 258)

Tasks

Like Critical Reasoning 18, there are no specific exercises for this one. Copi did however invite the interested reader to attempt to provide proofs for some of the theorems and derived rules that he did not set out explicitly.

Reference

COPI, I.M. (1979) *Symbolic Logic,* 5th Edition. Macmillan: New York

GRAHAM, R. et al. (1994) *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley: Reading, MA