Critical Reasoning 16 - Deductive Systems

A Young Jules Henri Poincaré (1854–1912)

"Science is built up with facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house."

(Henri Poincaré)

Copi (1979) begins his shortest, yet one of his conceptually most important chapters on deductive systems with the above quotation by the French polymath Henri Poincaré (at right). And what is true for Science, in this sense, is also true for Mathematics and Logic. A list of facts or true propositions is no more a science than a list of axioms and theorems is a formal system - for *Science* is *organised knowledge*. Moreover, the manner of organisation is important. We want knowledge to be organised *systematically* in such a way that their *interrelations* are manifest.

According to Copi:

> All knowledge that we possess can be formulated in propositions, and these propositions consist of terms. In any science, some propositions can be deduced from or proved on the basis of other propositions. For example, Galileo's laws of falling bodies and Kepler's laws of planetary motion are all derivable from Newton's more general laws of gravitation and motion. The discovery of these deductive interrelationships was an exciting phase in the development of the *science* of physics. Thus one important relationship among the propositions of science is deducibility. Propositions that embody knowledge about a subject become a *science* of that subject when they are arranged or ordered by displaying some of them as conclusions derived from others. (p. 157)

Some of the terms of propositions can themselves can be defined in terms of others. Copi uses the examples of *density* as *mass per unit volume* or *acceleration* as *rate of change in velocity* and *velocity* itself, which can in tern be defined as *rate of change of displacement*. Such definitions of terms by means of others reveals some of the interrelation among propositions within a common subject "… just as deductions integrate its laws or statements." (*l.c.*)

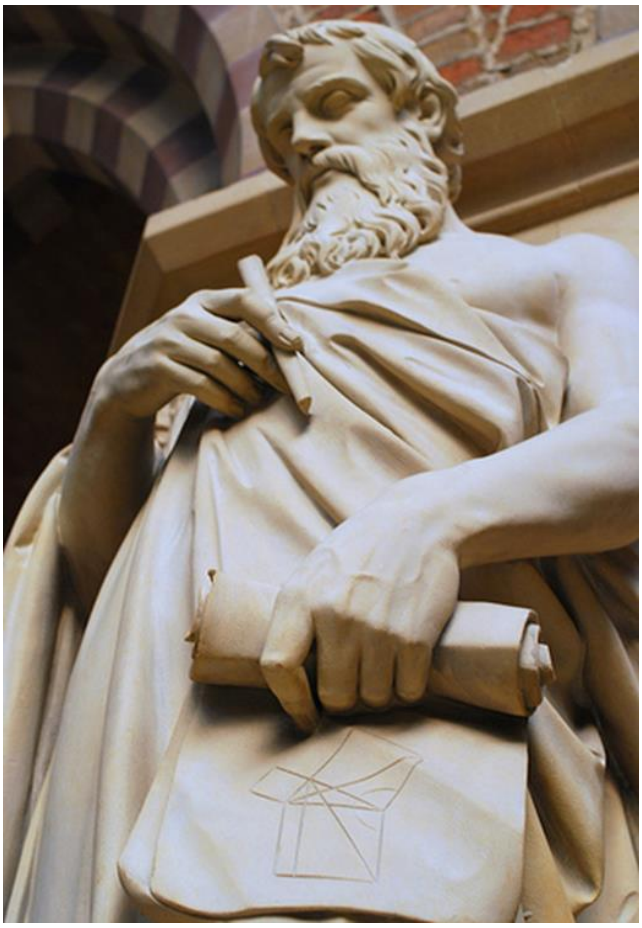
Although definition and induction are important to Science an "ideal science" in which all propositions are proved by deducing them from others and all terms are defined, would be "ideal" only in the negative sense. Since all terms have to be defined by means of other terms *within the system*, whose meanings must be understood beforehand, some definitions *within such a system* must necessarily be circular. Copi suggests the example of looking up the word "large" in a pocket dictionary only to find it defined as *big*, which in turn is defined as *large*. Similarly deductions can only derive their conclusions based on premises that have already been verified by other proofs. Either way, this will entail an infinite regress of proofs or proofs that are circular, neither of which are acceptable.[1] Therefore *within a system* that constitutes a science, not all terms can be defined

[1] Recall that this situation or something very like this was one of the premises of Aquinas' Cosmological Argument.

and not all propositions can be proved. It is not necessarily the case that a particular term cannot be defined or that a particular proposition cannot be proved, only that they cannot *all* be defined or *all* be proved *within a system*. According to Copi, an *ideal* system in the positive sense would be "one in which a minimum number of propositions suffices for the deduction of all the rest and a minimum number of terms suffices for the definition of all the others. This ideal of knowledge is described as a **deductive system**." (p. 158)

Euclidian Geometry

Although the study of geometry preceded Euclid by centuries, he was the first to present it as a systematised body of knowledge or science that is still taught in high school today. Although the Egyptians and Babylonians, thousands of years earlier, had used geometrical knowledge in the design of their pyramids and temples as well as the development of calendrical systems based on astronomical observations, such knowledge, according to Copi, "consisted of a collection or catalog of almost wholly isolated facts… a mere list of useful empirical rules-of-thumb for surveying land or constructing bridges or buildings…" (p. 159)



Euclid of Alexandria (fl. 300BC): Greek Mathematician and Father of Geometry. Seen here from below in the Oxford University Museum of Natural History with a Scroll Depicting Pythagoras' Theorem.

The Greeks, beginning with Pythagoras (6th century BC) and later with Euclid (3rd century BC), were the first to introduce order into the subject as a system. In Euclid's *Elements* all geometrical propositions are arranged in order, beginning with axioms, definitions and postulates, continuing with theorems deduced from the initial propositions. (*l.c.*) The first four definitions are:

Defn. 1: A point is that which has no parts.

Defn. 2: A line is a breadthless length.

Defn. 3: The extremities of lines are points.

Defn. 4: A straight line lies equally with respect to the points on itself.

Copi observes that although Euclid defines the terms "point" and "line" in the first two definitions he does not define the terms "part", "length" and "breadth" *used* in these definitions. Therefore the use of a defined term is logically only a matter of convenience since, theoretically every proposition that contains a defined term can be translated into one that contains only undefined terms. (p. 159 - 160) Consider Euclid's five postulates (or axioms):

Let the following be postulated:

P1: To draw a straight line from any point to any point.

P2: To produce [extend] a finite straight line continuously in a straight line.

P3: To describe a circle with any centre and distance [radius].

P4: That all right angles are equal to one another.

P5: That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (**parallel postulate**)

The *Elements* also includes the following five "common notions" which are essentially attributes of arithmetic relations (see Critical Reasoning 14):

CN1: Things that are equal to the same thing are also equal to one another. (**transitivity of equality**)

CN2: If equals are added to equals, then the wholes are equal. (**additivity of equality**)

CN3: If equals are subtracted from equals, then the remainders are equal. (**subtractivity of equality**)

CN4: Things that coincide with one another are equal to one another. (**reflexivity**)

CN5: The whole is greater than the part. (Likely definition of "greater than")

Although Euclid divided his unproven propositions, classifying some as *axioms* and others as *postulates*, the distinction has fallen away today and the terms may be used more or less interchangeably. Axioms (or postulates) are required by every deductive system in order to avoid circularity or "vicious" regression. They may be held to be self-evident or, if proved, then proved outside the system or extra-systematically. (p. 160)

According to one conception of Euclidian geometry, if a theorem follows logically from a set of self-evident axioms, then that theorem was considered to be "just as *true*" as the self-evident axioms from which it was derived. According to the modern view however, axioms are not held to be self-evidently true, only assumed to be true, rather than proved within a system. This view grew out of historical developments within geometry and physics.

Consider the famous "parallel postulate": As early at the 5thcentury A.D. the Neoplatonic philosopher Proclus "The Successor" wrote of it: "This ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties…" (Heath, 1926 p. 202) According to Copi, "although its *truth* was not questioned, its *self-evidence* was denied, which was deemed sufficient reason to demote it from its exalted position as an axiom to the more humble status of a mere theorem." (p. 161) Today however we know it cannot be proved. (Lewis, 1920)

The parallel postulate is in fact independent of the other Euclidian postulates and there are alternative branches of non-Euclidian geometry, such as those of Lobachevsky and Riemann, which do not rely on the truth of the parallel postulate in general, although there is one special case in which it is true, namely space, or rather space-time, with zero curvature. All indications however point to the fact that we inhabit a universe in which the geometry of space-time is non-Euclidian and we know from Einstein's theory of General Relativity that the mere presence of massive objects causes space-time to be warped locally around them.

According to Copi however, the truth or falsehood of the parallel postulate, or indeed any postulate (or axiom) "is a purely *external* property of any deductive system… It is no doubt important, to the extent that a deductive system is ordered *knowledge.* [However,] when we concentrate our attention on the system as such, its order is its more important characteristic." In this logical sense then, "a deductive system can be regarded as a vast and complex argument. Its premises are the axioms, and its conclusion is the conjunction of all theorems that are deduced". What matters for a deductive system, again from the logical perspective, is the validity of the inferences, not the veracity of the premises. If the inferences that lead to theorems are valid arguments then we can expect theorems to derive necessarily from axioms *if* they are true. Therefore, when we abstract away from the extra-systematic explanation of a deductive system's terms, the question of truth or falsity becomes irrelevant - it is the cogency that matters. (p. 162)

Formal Deductive Systems

There are several motivations for wanting to formalise a system, often to avoid lapses in rigor that come with being too familiar with one's subject. According to Copi, the mistake in Euclid's very first proof is a case in point. Even as non-geometers for example, we all have our private notions about the meanings of "point", "line", "plane", "space" *etc.* which entail unstated assumptions. When these are included in the development of a deductive system they render them less than rigorous.

In order to circumvent this, contemporary mathematicians abstract away from our familiar notions, replacing them with arbitrary symbols (mostly Greek and Roman letters). We have already been doing something similar since Critical Reasoning 05 by symbolising an argument and then checking its validity based on this symbolised form. By the use of primitive or undefined terms, which include uninterpreted symbols, mathematicians and logicians can develop ***formal*** **deductive systems** whose "postulates are not propositions at all, but mere formulas". (p. 163)

Of course, deductive relationships can exist among mere formulae but because they are unencumbered by our private notions or unconscious biases, proof of theorems involving them can proceed without the question of their truth of falsehood ever having to arise. After all, they are symbols, not propositions. It is possible therefore, to give the symbols differing interpretations. "And since the theorems are formal consequences of the axioms, any interpretation of the arbitrary symbols that makes the axioms true will necessarily make the theorems true, also." (*l.c.*)

So if we hold off on interpreting the undefined terms of a deductive system until after its theorems have been derived we can achieve, not only rigor, but also greater generality because we may be able to find alternative interpretations for the terms and applications for the system being formulated. Copi points to one such example in which different interpretations of the arbitrary primitive symbols of one deductive system lead to the theory of real numbers on the one hand, or to the theory of points on a straight line, which underpins that of analytical geometry. In Critical Reasoning 18 we will see how the formal deductive system of Class Algebra is also a Propositional Calculus. (p. 163 - 164)

Besides the arbitrary symbols, axioms and theorems of formal deductive system require nothing more than logical terms and connectives such as "if… then", "and", "or", "not", "all", and "are" as well as "sum", "product" and numerical symbols, if the system is intended for arithmetic. (p. 164) We have already seen in Critical Reasoning 05 how logical connectives can be defined by means of truth tables alone.

Attributes of Formal Deductive Systems

Anyone who wishes to set up a formal deductive system usually has some particular interpretation "in mind". *I.e.* they want to setup a system to express knowledge about a certain subject. Once it is set up, the system is said to be **expressively complete** *with respect to that subject matter* if it can adequately express or formulate all the propositions it is intended to express. Note that question of expressive completeness has no bearing on whether the propositions so expressed are either true or provable - that is another matter. (p. 164)

If a system has two or more formulae which are either axioms or can be proved as theorems within the system but that contradict one another or are the denial of each other, then such a system is said to be **inconsistent**. The problem with a genuine logical contradiction (even just one of them) is that everything follows, including every other possible contradiction. Known as ***ex falso quodlibet*** for "from falsehood, anything" or ***ex contradictione quodlibet segitur*** for "from contradiction, anything follows"; this valid argument can be trivially proved as follows: Let $p \bullet \sim p$ represent a contradiction and let q represent any proposition whatsoever, then

1. $p \bullet \sim p$ Contradiction
2. p 1 Simp.
3. $\sim p$ 1 Simp.
4. $p \vee q$ 2 Add
5. q 4, 3 D.S.

Emil Leon Post (1897 - 1954): Polish born American Mathematician and Logician

Hence, another way of defining consistency according to the **Post criterion for consistency** (after the Polish-American mathematician and logician Emil Leon Post) is as follows: "Any system is consistent if it contains (that is, can express) a formula that is not provable as a theorem within the system." (p. 164) Because the presence of a contradiction within a system would render every proposition a provable theorem, Post's criterion for consistency excludes such systems from the class of consistent systems.

Obviously an inconsistent system is useless to humans and machines alike and cannot serve as a systematisation of knowledge. However, searching for and failing to find a theorem and its negation within a system does not guarantee its consistency either, if only for lack of ingenuity. Short of an exhaustive check of every theorem (and there may be an infinite number), a formal deductive system can be proved to be consistent if an interpretation can be found that makes all the axioms true, for then all the derived theorems will also be true. (p. 165)

On the other hand, an axiom of a formal deductive system is said to be **independent** (or exhibit independence) if it cannot be derived as a theorem from the other axioms of the system. Axioms that are not independent are not logically “bad” or “fatal” in the way that inconsistencies are; however they are aesthetically unappealing and unparsimonious because, in the interests of economy, one *ought* to assume the least number of axioms necessary for the development of a formal system. Axioms that are not independent in this way is said to be **redundant**. (*l.c.*)

Similarly, trying and failing to find that an axiom is not derivable from the rest does not prove that it is independent, if again only for lack of ingenuity. Any particular axiom however can be proved to be independent of the others if an interpretation can be found on which the axiom in question is false while the remaining axioms are true. The roundabout reason being that, if such an axiom were derivable from the others, making them true would also make it true. If the same sort of interpretation can be found for each axiom in turn (*i.e.* on which each axiom in question is false while the remaining ones are true) then the set of all axioms can be said to be independent. (*l.c.*)

As we have alluded to above, an expressively complete system is one in which any desired proposition, with respect to the subject matter, can be expressed as a formula within the system. On the other hand a **deductively complete** system is, loosely speaking, one in which all desired formulae (or their negation) can be proved within it. If we have constructed the system in such a way that, on a given interpretation, all the true formulae express propositions, with respect to the subject matter, then we may also, loosely speaking, call that system deductively complete. (*l.c.*)

In an inconsequential way, an inconsistent system will be complete because any desired formula can be proved within it, but given their utter futility we need not be concerned with them. (p. 165 - 166)

Copi explains a further conception of completeness along the following lines:

> Any formal deductive system will have a certain collection of special undefined or primitive terms. Any terms that are definable within the system are theoretically eliminable, that is they are replaceable in any formula in which they occur. They can be so replaced by the sequence of undefined terms by means of which they were first defined. We shall ignore defined terms for the present. All formulas that contain no terms other than these special undefined terms (and logical terms) are expressible within the system. We may speak of the totality of undefined terms as the *base* of the system. The formulas expressible in the system are formulas constructed *on that base*. In general, the totality of formulas constructed on the base of a given system can be divided into three groups. First is the group of all formulas that are provable as theorems within the system. Second is the group of all formulas whose negations are provable within the system and third, the group of all formulas such that neither they nor their negations are provable within the system. For *consistent* systems, the first and second groups are *disjoint* or *mutually exclusive*, that is, they have no formulas in common. Any system whose third group is empty, that is, contains no formulas at all, is said to be *deductively complete*. An alternative way of phrasing this sense of completeness is to

> say that every formula of the system is such that either it or its negation is provable as a theorem. (p. 166)

Another definition of completeness that is entailed by the above is this: "… a deductive system is complete when every formula constructed on its base is either a theorem or else its addition as an axiom would make the system inconsistent." (*l.c.*) The first part of this definition takes care of formulae that are provable as theorems within the system. The second part takes care of those whose *negations* are provable, hence the reason why the system would be rendered inconsistent by their addition as axioms. The use of the conjunction "or else" here must be understood in the sense of the exclusive or (see "XOR" Critical Reasoning 05) to mean that none of the formulae referred to in the second part of the definition are also among those referred to in the first part. At any rate "or else" is not used inclusively in natural language.

Although completeness is the most prized attribute of any non-trivial formal deductive system, there are numerous systems that are incomplete and yet highly interesting and informative. Euclidian geometry (without its parallel postulate) is such an example. According to Copi, "the parallel postulate is, itself, a formula constructible on the base of the Euclidian system. Yet, neither it, nor its negation, is deducible from the other postulates." In spite of, or rather because of this, modern Cosmologists have been able to deduce properties of space independently of whether it is Euclidian (flat) or non-Euclidian (curved) or at least they have been able to discover their *common features*. (p. 166)

Logistic Systems

Though consistency and completeness and are desired attributes of any deductive system, none can be developed that lack rigour. In this sense a system is said to be **rigorous** if no formula is asserted to be a theorem unless it is *logically entailed* by its axioms. As with formal deductive systems generally, a system may be abstracted away from our familiar notions, replacing them with arbitrary symbols as undefined or primitive terms and then developed formally. Listing all the undefined terms and stating the axioms from which theorems are to be derived will help in deciding which formulae are to be considered theorems and which are not. (p. 166 - 167)

However even this does not guarantee rigour because such systems "assume logic, in the sense that their theorems are supposed to follow *logically* from their axioms." Therefore we cannot achieve complete rigor until the concept of *logical proof* or *logical derivation* can also be specified precisely. This would include "a list of valid argument forms or principles of valid inference," even though it would be unsystematic and practically impossible to simply catalogue *all* of them. What is needed is that a "deductive system of logic, itself, must be set up… [which] will have deduction, itself, as its subject matter". (p. 167)

A system of this sort is known as a **logistic system** - one which may not contain the ordinary assumed meanings of terms of logic such as "and", "or", "not", "if… then" *etc.* These must be replaced by uninterpreted symbols. Moreover the rules of inference that such a system uses to deduce logical theorems from the axioms it assumes must be few and made explicit. (*l.c.*)

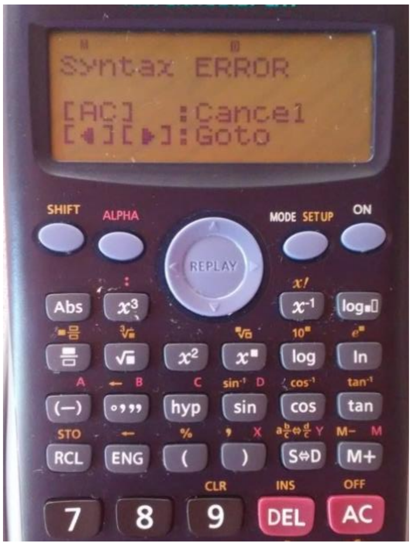
A logistic system must also specify a **syntax** by which to differentiate strings of symbols in a formal system that are to count as **well formed formulae** (*wffs*) from those that that are purely nonsensical. A *wff* may therefore be defined as a finite sequence of symbols from a given "alphabet" of a formal language that obeys the syntactic formation rules of the system. Anyone who has tried their hand at programming will be familiar with "syntax errors" that are returned when a line of code is encountered that violates the rules governing the composition of "meaningful" text in a programming language. The line:

110)GOSUB 20

in BASIC, for example, will encounter a syntax error because it contains an instruction indicating precedence that is not executable by the program interpreting the command. Similarly, the line:

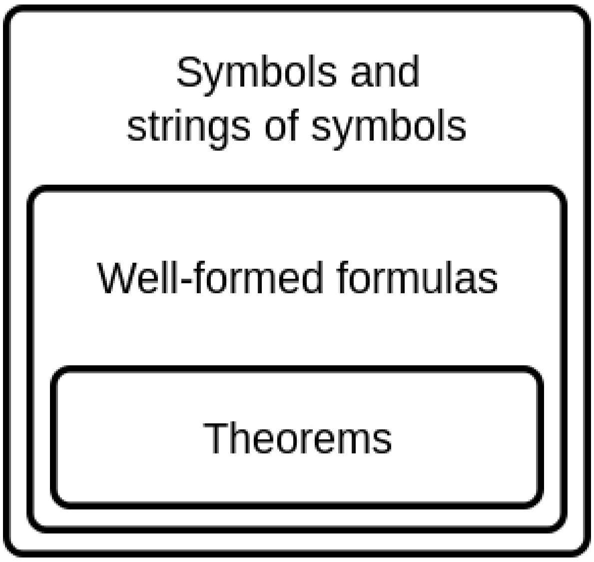
5. AB ⊃~

is not a well formed formula in an ordinary line of logical proof because the arrangement of symbols is incapable of being meaningful, irrespective of whether A or B are interpreted or not. A logistic system must go one step furtherer than simply recognising *wffs*. In the interests of rigor, it must also be able to do so in abstraction from the meaning of the symbols used for logical operators themselves. According to Copi: "In a logistic system, there must be a *purely formal* criterion for distinguishing well formed formulas from all others. To characterise the criterion as 'purely formal' is to say that it is *syntactical* rather than *semantical*, pertaining to the formal characteristics and arrangements of the symbols in abstraction from their meanings." (p. 168)



A High School Scientific Calculator Displays a Syntax Error after an Input String that is not a wff

Accordingly, such a criterion will be able to partition all symbols and strings of symbols into two mutually exclusive groups: one that contains all well-formed formulae (*wffs*) and another that contains the rest. (See right.) Although all *wffs* remain uninterpreted, they constitute a "language" of which, some will be chosen as axioms or postulates. (Not shown.) The rest of the *wffs* fall into two sub-groups: Those that are theorems and those that are not. Theorems are those that are "derivable" from the axioms or postulates *within the system*. (p. 169)



Three Syntactic Entities which may be Constructed from Formal Systems or Formal Languages - See text left. (*Image courtesy of Wikipedia: Syntax* (*logic*))

Because all *wffs* remain uninterpreted, validity which is a semantic notion, cannot be used as a criterion for "derivable" in this sense. Instead we must employ a purely formal or syntactical criterion of validity, in the sense that an argument is valid if and only if the *truth* of its premises entails the *truth* of its conclusion *within the system*. Syntactically valid arguments in this sense include not only the familiar ones with postulates as premises and theorems as conclusions but "may also have as premises *any wffs*, even those which are neither postulates nor theorems, and they may have as conclusions *wffs* that are not theorems." (p. 169)

Finally, the **interpretation** of a formal system consists of assigning meanings to symbols and truth values to sentences within the system. The study of such interpretations is known as **formal semantics** which is expressed in a metalanguage, usually trying to capture the pre-theoretic notion of entailment. A **metalanguage**, in turn, is a language used to describe another language, often called the **object language**. A metalanguage, may itself be a formal language, and as such itself is a syntactic entity, the structures of which can be described by a metasyntax. (Wikipedia: Syntax (logic) and Wikipedia: Metalanguage) "Of course," Copi remarks, "it is desired that any argument within the system which is syntactically 'valid' will become, when given its intended or 'normal' interpretation, a semantically valid argument." (*l.c.*)

Task

Can you identify the elements required for constructing a logistic system? Copi identifies five.

Feedback

On Poincaré's analogy of house built with stones, it is perhaps a little unfair to ask how such a house is to be constructed from the foundations up until one has actually first seen it accomplished. That we shall see in Critical Reasoning 21 when we examine Copi's presentation of a formalised propositional calculus. Until then here are the five elements he identifies that are required for constructing a logistic system:

1. a list of primitive symbols which, together with any symbol defined in terms of them, [that] are the only symbols that occur within the system;

2. a purely formal or syntactical criterion for dividing sequences of these symbols into formulas that are well formed (*wffs*) and those that are not;

3. a list of *wffs* assumed as postulates or axioms;

4. a purely formal or syntactical criterion for dividing sequences of *wffs* into 'valid' and 'invalid' arguments; and

5. derivatively from 3 and 4, a purely formal criterion for distinguishing between theorems and non-theorems of the system. (p. 169)

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