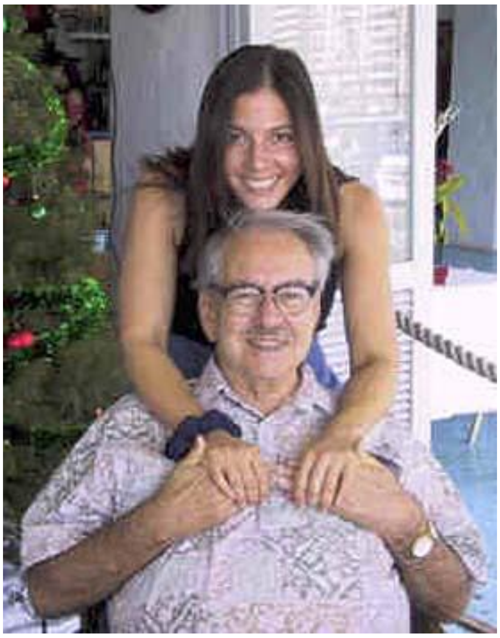
Critical Reasoning 11 - Predicate Logic

Up till now we have been treating sentences and the propositions they express as simple truth functional entities, whereas we know that they have an "inner logical structure" that cannot be adequately captured by a single letter, such as "C" for "A cat is on a mat." The calculus of sentences or **sentential** (or **propositional**) **calculus**, which has served us so well hitherto, cannot be applied to even the most famous argument of all:

> All humans are mortal.
> Socrates is human.
> Therefore, Socrates is a mortal.

As Copi points out, "The validity of such an argument depends upon the inner logical structure of the noncompound statements it contains." (p.63) The introduction of **predicates** that we can quantify allows us to express this "inner logic." It clear that Plato was aware of the distinction between **subjects and predicates** that we have been taught, even in primary school. Following convention we will use upper case letters to stand for predicates and lower case ones, *a* - *w* to stand for subjects. Thus the second premise above is represented: "Hs" where "s" is the subject, "Socrates" and "H" is the predicate, "is human." Similarly the conclusion above can be represented: "Ms" where ''s" is the same subject, "Socrates" and "M" is the predicate, "is mortal."



Proof positive that there is a human face to logic: Irving Marmer Copi (born Copilowish 1917 - 2002) with daughter Margaret. Bertrand Russell, Copi's teacher at the University of Chicago described him in his biography as one of his most impressive students. Educated in Mathematics and Philosophy, Copi devoted his life to study, teaching, writing and research. His bibliography includes 160 books and articles, with works translated into 10 languages. Growing up with such a brilliant father had its challenges, said Margaret, a Honolulu psychiatrist. "He would demand that we support our assertions" she said. "He would change a decision if reasons were convincing." (Honolulu Star-Bulletin - Obituary, Sunday, 1st September 2002)

A predicate can be any (affirmative) singular proposition that states that the individual referred to by its subject term has the attribute designated by its predicate term. These are usually expressed by adjectives but may also be expressed by certain nouns or even verbs. Subjects, on the other hand, can be individuals, things, places, nations, planets, stars and so on, and are therefore expressed by nouns. (*loc. cit.*)

Because many singular propositions can have the same predicate such as "Ha" for "Aristotle is human," "Hb" for "Belinda is human," "Hc" for "Charlie is human" and so on, we can use one of the letters: x,y or z as **individual variables**. While "Ha", "Hb" and "Hc" are true "Hx" is neither true nor false. In fact “Hx” is not even a proposition; rather it is a **propositional function** with x playing the role of a “place marker” in the same way that we use x’s in maths as place holders for numbers, angles and what have you.

When we substitute an **individual constant** (*e.g.* “a”, “b”, “c” *etc.*) for the individual variable “x” of a propositional function (*e.g.* “Hx”) the resulting singular proposition is said to be a **substitution instance** of the propositional function in question. Both singular propositions and propositional functions can be negated in the usual way, thus: “$\sim Ha$” symbolises “Aristotle is not mortal,” while “$\sim Hx$” negates whatever substitution instances of the propositional function there may be.

The first premise above is a general proposition. Other examples of general propositions are: “Everything is mortal,” “Nothing is mortal,” “Something is mortal,” and so on. For these we require logical **quantifiers**. These are denoted by a symbol (including a variable) that indicates the degree of generality of an expression. The two that concern us are the universal quantifier “∀” for “all” and the existential quantifier “∃” for “there exists.” Thus, the universal quantifier with its variable: “$(\forall x)$” reads “for all x…,” or “given any x…,”while the existential quantifier with its variable: “$(\exists x)$” reads “there exists an x such that….”

Using the universal quantifier with a propositional function such as “Mx” for “x is mortal,” allows us to represent the universal statement: “All x’s are mortal” as:

$$(\forall x)Mx$$

Similarly, using the existential quantifier with the same propositional function, “Mx”, allows us to represent the existential statement: “There exists an x such that x is mortal” as:

$$(\exists x)\ Mx$$

Copi (p. 65) suggests several paraphrasals of the same existential statement as:

Something is mortal.
There is at least one thing that is mortal.
There is at least one thing such that it is mortal.

Logically they all express the same proposition.

Universally and existentially quantified propositional functions are themselves truth functional: Universally quantified propositional functions are true if and only if all their substitution instances are true. On the other hand an existentially quantified propositional function is true if at least one substitution instance of it is true. Thus, “Everything is mortal” is true if and only if every last thing *in the Universe* is mortal; hence the term “*universal* quantification.” On the other hand, “Something is mortal” is true if *there exists* even one thing that is mortal; hence the term “*existential* quantification.”

Of course, so long as there exists at least one individual, (empty universes don’t count) then, if a universally quantified propositional function is true, then so is the same existentially quantified propositional function. *E.g.* if everything is mortal then there exists an individual such that it is mortal. Clearly things don’t work the other way round: the existence of *a* mortal individual does not imply the existence of *all* mortal individuals.

Universal or existential quantification of **negated propositional functions** reveals a further relation. Consider the following logically equivalent pairs of statements:

All things are not mortal: $(\forall x){\sim}Mx \equiv$ Nothing is mortal: ${\sim}(\exists x)Mx$

Something is not mortal: $(\exists x){\sim}Mx \equiv$ Not all things are mortal: ${\sim}(\forall x)Mx$

These show that, as Copi puts it rather tersely, "the negation of the universal (existential) quantification of a propositional function is logically equivalent to the existential (universal) quantification of the new propositional function that results from placing a negation symbol in front of the first propositional function" (*l.c.*) A loose and ready way of remembering the equivalence of negated propositional functions is to just "swap the quantifier and take the 'not' inside."

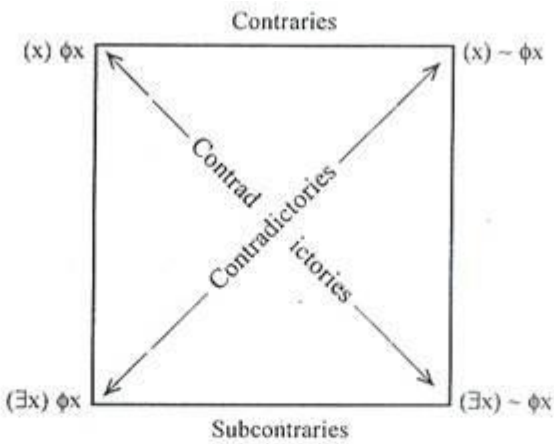
Furthermore, if a universally quantified propositional function is true then the existentially quantified negation of the same propositional function must be false and *vice versa*. Similarly, if an existentially quantified propositional function is true then the negation of the universally quantified same propositional function must be false and *vice versa*. In short they are **contradictories**. Consider:

All things are mortal: $(\forall x)Mx$ *vs*. Something is not mortal: $(\exists x){\sim}Mx$

Something is mortal: $(\exists x)Mx$ *vs*. Not all things are mortal: ${\sim}(\forall x)Mx$

If one of these pairs is true then other must be false and *vice versa*.

Copi (p. 66) illustrates the relations that obtain between universal and existential quantification by means of the square array at right, using the upper case Greek letter Φ (phi) to represent any attribute or symbol whatsoever. The contradictories above can be found at opposite diagonals. **Contraries**, where both may be false but cannot both be true, occupy the upper row, while **sub-contraries**, where both may be true but cannot both be false, occupy the lower row. On each vertical meanwhile, the proposition above implies the one below.



Multiply General Propositions

The first premise of our famous argument "All humans are mortal" is a universally quantified expression, however because it includes the conditional: "*if* something is human *then* it is mortal" it encompasses the complex propositional function "$Hx \supset Mx$" which has the same subject terms. The **scope of quantification** is very important and is indicated by parentheses: ( ), [ ], [( )] and so on. Thus, "All humans are mortal," which is an instance the **universal affirmative**, is symbolised:

$$(\forall x)(Hx \supset Mx)$$

This can variously be read as: "For all x, if x is human then x is mortal"; "Given any individual, if that individual is human, then that individual is mortal"; "For any x, x is human $\supset x$ is mortal" and so on; all of which express the same complex proposition. Notice that the parentheses indicate that here the scope of quantification encompasses the entire complex propositional function "$Hx \supset Mx$". Failing to take cognisance of the scope of quantification can result in in unintended expressions such as:

$$(\forall x)Hx \supset (\forall x)Mx$$

which reads "everything is human implies everything is mortal," which is certainly not the sense conveyed by the first premise.

It is also possible to incorporate negations into one or more of the terms bound by a quantifier in multiply general propositions. The **universal negative** "All humans are not mortal," is one such example and is symbolised:

$$(\forall x)(Hx \ \supset \sim Mx)$$

Whereas universally quantified expressions usually involve a "⊃" as the main connective, existentially quantified expressions usually involve an "and". Thus, an instance of the **particular affirmative** "something is both human and mortal" is symbolised as:

$$(\exists x)(Hx \bullet Mx)$$

Alternative paraphrasals include:

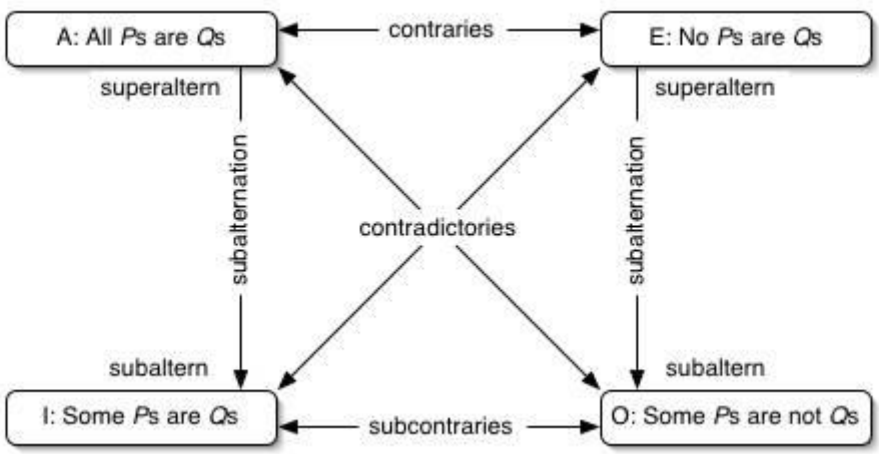
There is at least one thing that is human and mortal.
There is at least one x such that x is human and x is mortal.

It is also possible to negate one or more of the terms bound by an existential quantifier in multiply general propositions. In this way, for example, an instance of the **particular negative** "something is both human and not mortal" may be symbolised as:

$$(\exists x)(Hx \bullet \ {\sim}Mx)$$

Just as there are a set of relations among the singly general propositions depicted in the square array above, so there exists a isomorphic set of relations among the four, basic, named standard multiply general propositions listed and depicted below. Here "*P*"s and "*Q*"s stand for propositional functions, while the letters "A," "E," "I," and "O" are a carry-over from the medieval codification of logic in Latin and are retained merely by convention.

Universal Affirmative:	$(\forall x)(Px \supset Qx)$	A: All *P*s are *Q*s
Universal Negative:	$(\forall x)(Px \supset {\sim}Qx)$	E: No *P*s are *Q*s
Particular Affirmative:	$(\exists x)(Px \bullet Qx)$	I: Some *P*s are *Q*s
Particular Negative:	$(\exists x)(Px \bullet {\sim}Qx)$	O: Some *P*s are not *Q*s



The expressions A, E, I and O do not exhaust all forms of multiply general propositions, many of which involve quantification of more complex propositional functions and/or multiple or nested quantification. Consider the following examples by Copi (p. 69) in which the same rules of precedence as learned in Critical Reasoning 05 apply.

E.g. The general proposition "All members are either parents or teachers," is symbolised as:

$$(\forall x)[Mx \supset (Px \lor Tx)]$$

As Copi is at pains to point out this expression "does *not* mean same as 'All members are parents or all members are teachers,'" which would be symbolised:

$$(\forall x)(Mx \supset Px) \lor (\forall x)(Mx \supset Tx)$$

This sort of mistake is common enough and usually involves some kind of misunderstanding or ambiguity about the scope of the quantifier.

Meanwhile the general proposition "Some Senators are either disloyal or misguided," is symbolised:

$$(\exists x)[Sx \supset (Dx \lor Mx)]$$

Copi demonstrates that the proposition "Apples and bananas are nourishing," can be correctly symbolised in two ways, "either as a the conjunction of two A propositions,

$$[(\forall x)(Ax \supset Nx)] \bullet [(\forall x)(Bx \supset Nx)]$$

or as a single noncompound general proposition,

$$(\forall x)[(Ax \lor Bx) \supset Nx]$$

However the expression: $(\forall x)[(Ax \bullet Bx) \supset Nx]$ means something quite different, to the effect that "anything that is *both* an apple *and* a banana is nourishing," which is not the intended sense - could an apple ever simultaneously be a banana?

Unfortunately there is no mechanical method or rules for moving between natural language and the predicate calculus. As the examples above demonstrate one must get at the sense of a proposition rather than its syntax in representing general (and other) propositions. Copi's exercises on pages 69 - 71 provide further opportunity to practice doing so.

Quantification Rules and Formal Proofs of Validity

By now we would have no trouble representing the Socrates argument on page 1 as:

$1. (\forall x)(Hx \supset Mx)$
$2. Hs \; / \therefore Ms$

However we are not able to prove it using our existing nine rules of inference and ten rules of replacement alone. There are an additional four rules of inference governing quantification that allow us to move between general and particular propositions, drawing inferences along the way. In section 4.2 Copi introduces four preliminary quantification rules only to revoke them in section 4.5 and replace them with four more rigorous versions. In the interests of expediency we will proceed directly with the more rigorous formulations. These however require that we extend our definition of validity and introduce more meticulous notation.

To begin with we must distinguish between **free** and **bound variables**: An individual variable is **bound** if it is either part of, or lies within the scope of a quantifier. On the other hand, an individual variable is **free** if it is neither part of nor lies within the scope of a quantifier.

E.g. All the variables (x's) in the first premise of the Socrates argument are bound by the universal quantifier, whereas the last "x" in the expression:

$(\forall x)(Hx) \supset Mx$

is free because it lies outside of the scope of the universal quantifier. Similarly the "y" in the expression:

$(\exists x)(Hx \bullet My)$

is free because it is not bound by the existential quantifier "$(\exists x)$" which binds x's alone, even though it lies within the same bracket as the bound variable "x."

This distinction is important when we want to move between the general and particular and *vice versa*. One complication arises when we free certain variables bound by quantifies and end up with propositional functions, which can be neither true nor false and can therefore not form part of a formal proof of validity in the usual sense. A parallel difficulty arises when we want to bind free variables that are part of propositional functions that can be neither true nor false.

According to Copi (p. 90) we can extend the concept of validity to include propositional functions that result from propositions or propositional functions, so long as *every* substitution instance of free variables by individual constants results in a valid argument.

The following conventions are used in formulating the quantification rules:

- The upper case Greek letter "Φ" is used to refer to any propositional function whatsoever.
- The lower case letter "μ" (mu) is used to denote individual variables only.
- The lower case letter "ν" (nu) is may be used to denote either an individual variable or an individual constant.
- The expression "$\Phi\mu$" denote any proposition or propositional function.

- The expression "$\Phi\nu$" denotes the result of replacing every free occurrence of μ in $\Phi\mu$ by ν provided that if ν is a variable it must occur free in $\Phi\nu$ wherever μ occurs free in $\Phi\mu$.
- If $\Phi\mu$ contains no free occurrence of μ then $\Phi\mu$ and $\Phi\nu$ are identical.
- If μ and ν happen to be identical then so are $\Phi\mu$ and $\Phi\nu$.

According to Copi (p. 92) adopting these conventions helps to prevent unwanted (*i.e.* invalid) inferences from being allowed by the following quantification rules:

Universal Instantiation (UI)

$\underline{(\forall\mu)\Phi\mu}$
$\therefore \Phi\nu$

Universal instantiation allows us to free a particular instance such as "$Hs \supset Ms$" from the universal proposition "$(\forall x)(Hx \supset Mx)$" by replacing every bound instance of the variable "x" with the constant "s". This provides the essential step needed in proving the famous Socrates argument, thus:

1. $(\forall x)(Hx \supset Mx)$
2. Hs /$\therefore$ Ms
3. $Hs \supset Ms$ 1, UI
4. Ms 3,2 M.P.

There are however a couple of constraints on universal instantiation that prevent us from making invalid inferences. The first is that UI must be applied to a whole line at a time in a proof. If you need to apply it to part of a line and it is possible to simplify that line, then you should simply first. The second is that *all instances* of universally quantified variables must be instantiated. *E.g.* If there are two universally quantified variables in a proposition say, "x" and "x", you cannot instantiate the one with the individual constant say, "s" while leaving the other as variable.

Existential Generalisation (EG)

$\underline{\Phi\nu}$
$\therefore (\exists\mu)\Phi\mu$

As the name implies, existential generalisation allows us to generalise about the existence of entities from one or more particulars. If Socrates is human then it is a clear that *a* human exists. As with UI, EG must be applied to a whole line at a time in a proof. There are no further restrictions on EG that are not covered by the general conventions governing the use of $\Phi\mu$ and $\Phi\nu$ above.

Copi (p. 95) illustrates the use of both UI and EG in proving the following argument that we would otherwise not have been able to prove before:

All humans are mortal.
Therefore, if Socrates is human, then some humans are mortal.

which may be symbolised as:

$(\forall x)(Hx \supset Mx)$ /$\therefore$ $Hs \supset (\exists x)(Hx \bullet Mx)$

and proved as follows:

1. $(\forall x)(Hx \supset Mx)$ /∴ $Hs \supset (\exists x)(Hx \bullet Mx)$
2. Hs
3. $Hs \supset Ms$ 1 UI
4. Ms 3,2 M.P.
5. $Hs \bullet Ms$ 2,4 Conj.
6. $(\exists x)(Hx \bullet Mx)$ 5 E.G.
7. $Hs \supset (\exists x)(Hx \bullet Mx)$ 2-6 C.P.

In general, as Copi (*l.c.*) demonstrates, any argument of the form:

$(\forall \mu)\Phi\mu$

∴ $(\exists \mu)\Phi\mu$

is valid according to the following proof:

1. $(\forall \mu)\Phi\mu$ /∴ $(\exists \mu)\Phi\mu$
2. $\Phi\nu$ 1 UI
3. $(\exists \mu)\Phi\mu$ 2 EG

Existential Instantiation (EI)

The formulation of EI is a little trickier than the other quantification rules but for good reason. We don't want to allow for inferences of the following kind:

Something is square.
Something is round.
Therefore something is both square and round.

If we set out this argument in symbolic form we will be able to spot the fault.

1. $(\exists x)Sx$
2. $(\exists x)Rx$ /∴ $(\exists x)(Sx \bullet Rx)$
3. Sz 1 EI
4. Rz 2 EI (*wrong*)
5. $Sz \bullet Rz$ 3, 4 Conj.
6. $(\exists x)(Sx \bullet Rx)$ 5 EG

The problem lies with line 4. We have already instantiated something as square therefore we should prevent further instantiations of the same free variable. Before setting out his method Copi establishes the following equivalence (E):

$$(\forall \nu)(\Phi\nu \supset p) \equiv [(\exists \mu)(\Phi\mu) \supset p]$$

"where ν occurs free in $\Phi\nu$ at all *and only* those places that μ occurs free in $\Phi\mu$, and where p contains no free occurrence of the variable ν." (p. 96)

Suppose we have $(\exists \mu)(\Phi\mu)$ as a line in a proof and that $\Phi\nu$ will allow us to derive some proposition *p* that we are trying to prove. Then so long as *p* has no free occurrence of ν we can write down $\Phi\nu$ as an assumption as part of a conditional proof by which to prove p. Once we have our *p* the rule of conditional proof requires us to close off our assumption thus:

i. $(\exists\mu)(\Phi\mu)$
→ j. $\Phi\nu$ (where ν is a variable that has no occurrence in any preceding line)
.
.
.
k. p (which contains no free occurrence of ν)
k+1. $\Phi\nu \supset p$ j. - k. C.P.

But $\Phi\nu \supset p$ is not what we are after p is. This can be derived in just a few further lines:

k+1. $\Phi\nu \supset p$
k+2. $(\forall\nu)(\Phi\nu \supset p)$ k+1 UG
k+3. $(\exists\mu)(\Phi\mu) \supset p$ k+2 Equivalence (E) above
k +4. p k+3, i, M.P.

So long as we apply it to a whole line at a time, potentially any legitimate existential instantiation can be constructed corresponding to the form above. Accordingly, Cop's rule for EI can be stated as:

$(\exists\mu)\Phi\mu$
→ $\Phi\nu$
.
.
.
p
$\therefore p$

with the important proviso that ν is a variable that does not occur free either in p or any line preceding $\Phi\nu$. (p. 97)

The following is an example of the legitimate use of EI in a proof:

1. $(\forall x)[Fx \supset (Gx \bullet Hx)]$
2. $(\exists x)Fx$ /∴ $(\exists x)Gx$
→3. Fy
4. $Fy \supset (Gy \bullet Hy)$ 1 UI
5. $Gy \bullet Hy$ 3,4 M.P.
6. Gy 5 Simp.
7. $(\exists x)Gx$ 6 EG
8. $(\exists x)Gx$ 2, 3-7 EI

Note that the justification written to the right when closing off an EI consists of the number of the line consisting of $(\exists\mu)\Phi\mu$ followed by the hyphenated numbers of lines from $\Phi\nu$ to p.

We have already noted that the argument "proving" the existence of something that is both square and round is invalid. According to Copi's rules for EI, the reason that the error lies in line 4 below is because z occurs free in the preceding line, as shown.

1. $(\exists x)Sx$
2. $(\exists x)Rx$ /∴ $(\exists x)(Sx \bullet Rx)$
→ 3. Sz
→ 4. Rz (*wrong*)

As with all human endeavours, there are more was of doing something wrong than doing it right. The following is another illegitimate use of EI:

1. $(\exists x)(Fx \bullet Gx)$ /∴ $(\forall x)Fx$
→2 $Fx \bullet Gy$ (*wrong*)
3. Fx 2 Simp
4. Fx 1, 2-3 EI (*wrong*)
5. $(\forall x)Fx$ 4 UG

The problem in line 2, according to Copi, is that our ν ("y" in this case) does not occur free in $\Phi\nu$ ("$Fx \bullet Gy$" in this case) at all places that μ ("x" in this case) occurs free in $\Phi\mu$ ("$Fx \bullet Gx$" in this case). Therefore "$Fx \bullet Gy$" is not a legitimate $\Phi\nu$ for use in applying EI where $(\exists\mu)\Phi\mu$ is $(\exists x)(Fx \bullet Gx)$ in this case. (Paraphrased, p. 98)

On the other hand, simply letting the predicates "F" and "G" stand for something such as "…is a fruit" and "…is a grape," respectively, yields the following obviously fallacious argument:

There exists something that is a fruit and a grape.
Therefore all things are fruit.

Universal Generalisation (UG)

Copi's more complicated formulation of EI above allows for a somewhat easier statement of UG, applied to a whole line of proof, as:

$\Phi\nu$
∴ $(\forall\mu)\Phi\mu$

provided that ν is a variable that does not occur free either in $(\forall\mu)\Phi\mu$ or in any assumption within whose scope $\Phi\nu$ lies. (p.99)

The last proviso, that ν does not occur free within the scope within which $\Phi\nu$ lies, is necessary to prevent fallacies of the following kind:

Something is both red and square.
Therefore everything is red.

If we set out this argument in symbolic form we will be able to spot fault, as we did before.

1. $(\exists x)(Rx \bullet Sx)$ /∴ $(\forall x)Rx$
→2. $Rx \bullet Sx$
3. Rx 2 Simp.
4. $(\forall x)Rx$ 3 UG (*wrong*)
5. $(\forall x)Rx$ 1, 2-5 EI

The problem with line 4 above is that our ν (in this case "x") is free within the scope within which $\Phi\nu$ (in this case "Rx") lies. Meanwhile, the other requirement that ν not occur free in $(\forall\mu)\Phi\mu$ is intended to disallow fallacies of this sort:

All fire trucks are red.
Therefore, for any thing and any other thing, if one is a fire truck then the other is red.

Again, setting out this argument in symbolic form will allow us spot the fault.

1. $(\forall x)(Fx \supset Rx)$ /∴ $(\forall x)(\forall y)(Fx \supset Ry)$
2. $Fx \supset Rx$ 1 UI
3. $(\forall y)(Fx \supset Ry)$ 2 UG (*wrong*)
4. $(\forall x)(\forall y)(Fx \supset Ry)$ 3 UG

The problem with line 3 above is that our variable "x" occurs free in "$(\forall y)(Fx \supset Ry)$" in which only the "$y$" is bound.

Lest we spend more time on how not to universally generalise, here are two proofs that contain legitimate uses of UI.

a)

1. $(\forall x)(Nx \supset Ox)$ /∴ $(\forall x)(Px \supset [(\forall y)(Py \supset Ny) \supset Ox])$	
→ 2. Px	
→ 3. $(\forall y)(Py \supset Ny)$	
4. $Px \supset Nx$	3 UI
5. Nx	2, 4 M.P.
6. $Nx \supset Ox$	1 UI
7. Ox	6, 5 M.P.
8. $(\forall y)(Py \supset Ny) \supset Ox$	3-7 C.P.
9. $Px \supset [(\forall y)(Py \supset Ny) \supset Ox]$	2-8 C.P.
10. $(\forall x)(Px \supset [(\forall y)(Py \supset Ny) \supset Ox])$	9 UG

b.

1. $(\forall x)(Ax \supset Bx)$ /∴ $(\forall x)Ax \supset (\forall y)By$	
→ 2. $(\forall x)Ax$	
3. Ay	2 UI
4. $Ay \supset By$	1 UI
5. By	4,3 M.P.
6. $(\forall y)By$	5 UG
7. $(\forall x)Ax \supset (\forall y)By$	2-6 C.P.

Quantifier Negation

Finally there are four logical equivalences involving the negation of quantifiers that can be used in proofs. Abbreviated simply as QN they can be stated as:

$$(\forall \nu)\Phi\nu \equiv \sim(\exists \nu)(\sim\Phi\nu)$$

$$\sim(\forall \nu)\Phi\nu \equiv (\exists \nu)(\sim\Phi\nu)$$

$$(\forall \nu)(\sim\Phi\nu) \equiv \sim(\exists \nu)(\Phi\nu)$$

$$\sim(\forall \nu)(\sim\Phi\nu) \equiv (\exists \nu)(\Phi\nu)$$

According to Copi (p. 109) each of these can be proved in terms of the propositional function "Fx" for "$\Phi\nu$" since proof of these equivalences do not depend on any of the peculiarities of "Fx" or the variable "x". Recall the rule of Material Equivalence (Equiv.):

$$(p \equiv q) \equiv [(p \supset q) \bullet (q \supset p)]$$

In other words, the biconditional "$p \equiv q$" is equivalent to "p implies q and q implies p." So if we can set up two conditional proofs side by side, the one proving "$p \supset q$" and the other "$q \supset p$" we shall have proven the equivalence, thus:

→ p	→ q
…	…
…	…
q	q
$p \supset q$	$q \supset p$

Proving the first equivalence, $(\forall \nu)\Phi\nu \equiv \sim(\exists \nu)(\sim\Phi\nu)$ can be done in terms of "Fx" *i.e.*

$$(\forall x)Fx \equiv \sim(\exists x)(\sim Fx)$$

So, on the left we shall have to prove:
"$(\forall x)Fx \supset \sim(\exists x)(\sim Fx)$" and then, on the right: "$\sim(\exists x)(\sim Fx) \supset (\forall x)Fx$," thus:

→ 1. $(\exists x)(\sim Fx)$		→ 1. $\sim(\exists x)(\sim Fx)$	
→ 2. $\sim Fy$		→ 2. $\sim Fy$	
→ 3. $(\forall x)Fx$		3. $(\exists x)(\sim Fx)$	2 EG
4. Fy	3 UI	4. $\sim Fy \supset (\exists x)(\sim Fx)$	2-3 C.P.
5. $(\forall x)Fx \supset Fy$	3-4 C.P.	5. Fy	4,1 M.T, D.N.
6. $\sim(\forall x)Fx$	5,2 M.T.	6. $(\forall x)Fx$	5 UG
7. $\sim(\forall x)Fx$	1, 2-6 EI	7. $\sim(\exists x)(\sim Fx) \supset (\forall x)Fx$	1-6 C.P.
8. $(\exists x)(\sim Fx) \supset \sim(\forall x)Fx$	1-7 CP		
9. $(\forall x)Fx \supset \sim(\exists x)(\sim Fx)$	8 Trans., D.N.		

The remaining three equivalences of quantifier negation can be proven in a similar manner. Indeed Copi devotes the remainder of section 4.7 to demonstrating several "fairly obvious logical truths" using the same side-by-side technique.

More Proofs of Invalidity:

If an argument such as:

$(\forall x)[Fx \supset Gx]$
$(\exists x)Fx$
$\therefore (\exists x)Gx$

is valid, then it is valid for all possible universes of objects. Thus if a universe contains only one object ***a***, then it is valid for that universe. Similarly, if a universe contains exactly two objects ***a*** and ***b*** then it is also valid for that universe, and so on. If we can find at least one universe in which we can prove it invalid, then it is invalid.

For any given model containing exactly ***k*** objects, ***a***, ***b***, ***c***…,***k*** we may use the following logical equivalences:

$(\forall x)\Phi x \equiv (\Phi a \bullet \Phi b \bullet \Phi c \bullet \ldots \bullet \Phi k)$ and

$(\exists x)\Phi x \equiv (\Phi a \vee \Phi b \vee \Phi c \vee \ldots \vee \Phi k)$

to translate multiply general propositions into logically equivalent truth-functional propositions. (Copi, p. 106) Thus on a model of a universe of one object (represented as $\boxed{a}$) where $(\forall x)Fx \equiv Fa$

$$(\forall x)[Fx \supset Gx] \equiv (Fa \supset Ga)$$

and where $(\exists x)Fx \equiv Fa$

$$(\exists x)[Fx \bullet Gx] \equiv (Fa \bullet Ga).$$

Notice that there is no disjucnt (or) above because we are dealing with a model of single object universe. If something is quantified of x to exist then there is only one object on the model, a in terms of which it can be substantiated.

In the case of a model of a two object universe: $\boxed{a, b}$

$$(\forall x)Fx \equiv (Fa \bullet Fb)$$

$$(\forall x)[Fx \supset Gx] \equiv [(Fa \supset Ga) \bullet (Fb \supset Gb)]$$

$$(\exists x)Fx \equiv (Fa \vee Fb)$$

$$(\exists x)[Fx \bullet Gx] \equiv [(Fa \bullet Ga) \vee (Fb \bullet Gb)]$$

And so on for models of universes containing three or more objects. In general, we can prove an invalid argument invalid by translating it into logically equivalent truth-functional propositions in terms of models consisting of first one object, then two, then three and so on. As we saw in Critical Reasoning 09, each argument corresponds to a particular truth-functional proposition which we can test for contingency. If any proposition corresponding to an argument on any model turns out to be contingent, then the argument to which it corresponds will be invalid on that model and therefore invalid.

Consider the following two arguments from Copi (Ex II p. 82):

1.) All astronauts are brave. Jim is brave. Therefore Jim is an astronaut.

This argument may be symbolised as:

$(\forall x)[Ax \supset Bx]$
Bj
$\therefore Aj$

We now test this on a model representing a universe of just one object, $\boxed{j}$. Translating the argument into logically equivalent truth-functional propositions in terms of j we get:

$Aj \supset Bj$
Bj
$\therefore Aj$

This now corresponds to the truth function:

$$[(Aj \supset Bj) \bullet Bj] \supset Aj$$

Using the shorter truth table technique, we find that there is no contradiction in assigning a value of false under the major conditional, thus:

$$[(Aj \supset \; Bj) \bullet \; Bj] \supset \; Aj$$

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Therefore this truth functional proposition is contingent and the argument to which it corresponds on the model $\boxed{j}$ is invalid, therefore it is invalid.

5.) No kittens are large. Some mammals are large. Therefore no kittens are mammals.

This argument may be symbolised as:

$$(\forall x)[Kx \supset {\sim}Lx]$$
$$(\exists x)[Mx \bullet Lx]$$
$$\therefore (\forall x)[Kx \supset {\sim}Mx]$$

On the model $\boxed{a}$ we have:

$$Ka \supset {\sim}La$$
$$Ma \bullet La$$
$$\therefore Ka \; \supset {\sim}Ma$$

Now, the truth function to which this argument corresponds:

$$[(Ka \supset {\sim}La) \bullet (Ma \bullet La)] \supset (Ka \supset {\sim}Ma)$$

turns out to have a contradiction when we try to assign a valve of false under the major conditional, which means that it is a tautology. Therefore this argument is valid in a 1 member universe. However because we suspect that this argument is still invalid, we proceed to a model of a 2 member universe, $\boxed{a, b}$:

$$(Ka \supset {\sim}La) \bullet (Kb \supset {\sim}Lb)$$
$$(Ma \bullet La) \vee (Mb \bullet Lb)$$
$$\therefore (Ka \supset {\sim}Ma) \bullet (Kb \supset {\sim}Mb)$$

The truth function to which this argument corresponds is contingent; therefore the argument is invalid on this model, and so it is invalid.

This concludes the present section on predicate logic. In Critical Reasoning 14 we shall be concerned with the logic of relations. You are encouraged to attempt the tasks below by way of practice. All are straightforward, although some may require an awareness of subtlety in meaning, especially the translations.

Tasks

The following tasks, taken from Copi Ch. 4, involve translations and formal proofs involving multiply general monadic propositions. Note there may be more than one way of correctly symbolising a proposition or proving an argument valid.

Translations using the suggested notation (Copi, p. 88)

1. If anything is damaged, someone will be blamed.
   Dx: x is damaged. Px: x is a person. Bx: x will be blamed.

2. If anything is damaged, the tenant will be charged for it.
   Cx: x will be charged to the tenant.

3. If nothing is damaged, nobody will be blamed.

4. If something is damaged, but nobody is blamed, the tenant will not be charged for it.
5. If any bananas are yellow, they are ripe.
   Bx: x is a banana. Yx: x is yellow. Rx: x is ripe.

6. If any bananas are yellow, then some bananas are ripe.

7. If any bananas are yellow, then if all yellow bananas are ripe, they are ripe.

8. If all ripe bananas are yellow, some yellow things are ripe.

Proofs of validity (Copi, p. 103)

1. $(\forall x)(Ax \supset Bx)$
   $\therefore (\forall x)(Bx \supset Cx) \supset (Ak \supset Ck)$

2. $(\forall x)(Dx \supset Ex)$
   $\therefore Da \supset [(\forall y)(Ey \supset Fy) \supset Fa]$

3. $(\forall x)[Gx \supset (\forall y)(Hy \supset Iy)]$
   $\therefore (\forall x)Gx \supset (\forall y)(Hy \supset Iy)$

4. $(\exists x)Jx \supset (\exists y)Ky$
   $\therefore (\exists x)[Jx \supset (\exists y)Ky]$

5. $(\exists x)Lx \supset (\forall y)My$
   $\therefore (\forall x)[Lx \supset (\forall y)My]$

6. $(\forall x)(Nx \supset Ox)$
   $\therefore (\forall x)\{Px \supset [(\forall y)(Py \supset Ny) \supset Ox]\}$

7. $(\forall x)(Qx \supset Rx)$
   $(\forall x)(Sx \supset Tx)$
   $\therefore (\forall x)(Rx \supset Sx) \supset (\forall y)(Qy \supset Ty)$

8. $(\exists x)Ux \supset (\forall y)[(Uy \vee Vy) \supset Wy]$
   $(\exists x)Ux \bullet (\exists x)Wx$
   $\therefore (\exists x)(Ux \bullet Wx)$

9. $(\exists x)Xx \supset (\forall y)(Yy \supset Zy)$
   $\therefore (\exists x)(Xx \bullet Yx) \supset (\exists y)(Xy \bullet Zy)$

Feedback

Translations

1. If anything is damaged, someone will be blamed.

$(\forall x)[Dx \supset (\exists y)(Py \bullet By)]$

For all x, if x is damaged, then there exists a y such that y is both a person and will be blamed.

2. If anything is damaged, the tenant will be charged for it.

$(\forall x)(Dx \supset Cx)$

For all x, if x is damaged, then x will be charged to the tenant.

3. If nothing is damaged, nobody will be blamed.

$\sim(\exists x)Dx \supset (\forall y)(Py \supset \sim By)$ or $(\forall x)(\sim Dx) \supset (\forall y)(Py \supset \sim By)$

If there does not exist an x, such that x is damaged, then, for all y such that y is a person, y will not be blamed.

4. If something is damaged, but nobody is blamed, the tenant will not be charged for it.

$(\forall x)\{[Dx \bullet (\forall y)(Py \supset \sim By)] \supset \sim Cx\}$

For all x, if x is damaged and for all y, if y is a person then y is not blamed, then x will not be charged to the tenant.

Note: although the "if something" above initially suggest existential quantification, the phrasing "if… then" of the main clause tells us that the major operator is a "⊃" which is almost without exception under universal quantification. The nested conditional "$(\forall y)(Py \supset \sim By)$" meanwhile is necessary to stipulate that it is *persons* who are not blamed. If we omit this from our expression in favour of just "$\sim Bx$" we will be allowing for any entities that are not blamed.

5. If any bananas are yellow, they are ripe.

$(\forall x)[(Bx \bullet Yx) \supset Rx]$

For all x, if x is a banana and x is yellow, then x is ripe.

6. If any bananas are yellow, then some bananas are ripe.

$(\exists x)(Bx \bullet Yx) \supset (\exists y)(By \bullet Ry)$

If there exists an x, such that x is both a banana and yellow, then there exists a y, such that y is both a banana and ripe.

7. If any bananas are yellow, then if all yellow bananas are ripe, they are ripe.

$(\forall x)\{(Bx \bullet Yx) \supset [(\forall y)[(By \bullet Yy) \supset Ry] \supset Rx]\}$

Note: as in 4. above, "if any" initially suggests existential quantification, however the phrasing "if… then" of the main clause tells us that the major operator is a "⊃" and therefore under universal quantification. The nested conditional "$(\forall y)[(By \bullet Yy) \supset Ry]$" expresses the nested "then if all..." in the subordinate cause. Also note that because there are three conditionals involved it is especially important to avoid scope ambiguities by indicating the order of precedence by means of brackets. Any other placement and we shall be representing another proposition.

8. If all ripe bananas are yellow, some yellow things are ripe.

$(\forall x)[(Bx \bullet Rx) \supset Yx] \supset (\exists y)(Yy \bullet Ry)$

For all x, if x is both a banana and ripe then x is yellow implies that there exists a y such that y is both yellow and ripe.

Proofs of validity

1.

1. $(\forall x)(Ax \supset Bx)$ /∴ $(\forall x)(Bx \supset Cx) \supset (Ak \supset Ck)$	
→2. $(\forall x)(Bx \supset Cx)$	
3. $Ak \supset Bk$	1 UI
4. $Bk \supset Ck$	2 UI
5. $Ak \supset Ck$	3,4 H.S.
6. $(\forall x)(Bx \supset Cx) \supset (Ak \supset Ck)$	2-5 CP

2.

1. $(\forall x)(Dx \supset Ex)$ /∴ $Da \supset [(\forall y)(Ey \supset Fy) \supset Fa]$	
→2. Da	
→3. $(\forall y)(Ey \supset Fy)$	
4. $Da \supset Ea$	1 UI
5. $Ea \supset Fa$	3 UI
6. $Da \supset Fa$	4,5 H.S.
7. Fa	6,2 M.P.
8. $(\forall y)(Ey \supset Fy) \supset Fa$	3-7 CP
9. $Da \supset [(\forall y)(Ey \supset Fy) \supset Fa]$	2-8 CP

3.

1. $(\forall x)[Gx \supset (\forall y)(Hy \supset Iy)]$ /∴ $(\forall x)Gx \supset (\forall y)(Hy \supset Iy)$	
→2. $(\forall x)Gx$	
3. $Gx \supset (\forall y)(Hy \supset Iy)$	1 UI
4. Gx	2UI
5. $(\forall y)(Hy \supset Iy)$	3,4 M.P.
6. $(\forall x)Gx \supset (\forall y)(Hy \supset Iy)$	2-5 CP

4.

1. $(\exists x)Jx \supset (\exists y)Ky$ /∴ $(\exists x)[Jx \supset (\exists y)Ky]$	
→2. $Jx \supset (\exists y)Ky$	
3. $(\exists x)[Jx \supset (\exists y)Ky]$	2 EG
4. $(\exists x)[Jx \supset (\exists y)Ky]$	1,2-3 EI

5.

1. $(\exists x)Lx \supset (\forall y)My$ /∴ $(\forall x)[Lx \supset (\forall y)My]$
2. Lx
3. $(\exists x)Lx$ 2 EG
4. $(\forall y)My$ 1,3 M.P.
5. $Lx \supset (\forall y)My$ 2-4 CP
6. $(\forall x)[Lx \supset (\forall y)My]$ 5 UG

6.

1. $(\forall x)(Nx \supset Ox)$ /∴ $(\forall x)\{Px \supset [(\forall y)(Py \supset Ny) \supset Ox]\}$
2. Px
3. $(\forall y)(Py \supset Ny)$
4. $Px \supset Nx$ 3 UI
5. Nx 4,2 M.P.
6. $Nx \supset Ox$ 1 UI
7. Ox 6,5 M.P
8. $(\forall y)(Py \supset Ny) \supset Ox$ 3-7 CP
9. $Px \supset [(\forall y)(Py \supset Ny) \supset Ox]$ 2-8 CP
10. $(\forall x)\{Px \supset [(\forall y)(Py \supset Ny) \supset Ox]\}$ 9 UG

7.

1. $(\forall x)(Qx \supset Rx)$
2. $(\forall x)(Sx \supset Tx)$ /∴ $(\forall x)(Rx \supset Sx) \supset (\forall y)(Qy \supset Ty)$
3. $(\forall x)(Rx \supset Sx)$
4. $Rx \supset Sx$ 3 UI
5. $Sx \supset Tx$ 2 UI
6. $Rx \supset Tx$ 4,5 H.S.
7. $Qx \supset Rx$ 1 UI
8. $Qx \supset Tx$ 7,6 H.S.
9. $(\forall y)(Qy \supset Ty)$ 8 UG
10. $(\forall x)(Rx \supset Sx) \supset (\forall y)(Qy \supset Ty)$ 3-9 CP

8.

1. $(\exists x)Ux \supset (\forall y)[(Uy \vee Vy) \supset Wy]$
2. $(\exists x)Ux \bullet (\exists x)Wx$ /∴ $(\exists x)(Ux \bullet Wx)$
3. $(\exists x)Ux$ 2 Simp.
4. $(\forall y)[(Uy \vee Vy) \supset Wy]$ 1,3 M.P.
5. Ux
6. $(Ux \vee Vx) \supset Wx$ 4 UI
7. $Ux \vee Vx$ 5 Add
8. Wx 6,7 M.P.
9. $Ux \bullet Wx$ 5,8 Conj.
10. $(\exists x)(Ux \bullet Wx)$ 9 EG
11. $(\exists x)(Ux \bullet Wx)$ 3,5-10 EI

9.

1.	$(\exists x)Xx \supset (\forall y)(Yy \supset Zy)$ /∴ $(\exists x)(Xx \bullet Yx) \supset (\exists y)(Xy \bullet Zy)$	
→2.	$(\exists x)(Xx \bullet Yx)$	
→3.	$Xx \bullet Yx$	
4.	Xx	3 Simp.
5.	$(\exists x)Xx$	4 EG
6.	$(\forall y)(Yy \supset Zy)$	1,5 M.P.
7.	$Yx \supset Zx$	6 UI
8.	Yx	3 Com., Simp.
9.	Zx	7,8 M.P.
10.	$Xx \bullet Zx$	4,9 Conj.
11.	$(\exists y)(Xy \bullet Zy)$	10 EG
12.	$(\exists y)(Xy \bullet Zy)$	2,3-11 EI
13.	$(\exists x)(Xx \bullet Yx) \supset (\exists y)(Xy \bullet Zy)$	2-12 CP

Reference

COPI, I.M. (1979) *Symbolic Logic,* 5th Edition. Macmillan: New York

Acknowledgement: Several of the explanations and examples above have been compiled from lecture notes on symbolic logic presented by the University of Cape Town. The sections on EI and UG in particular are drawn from a hand-out on Quantification Theory provided by Professor Ian Bunting. The section on further proofs of invalidity was reconstructed from lecture notes on the topic by the same lecturer.

Unfortunately, Copi did not provide solutions to most of his exercises in *Symbolic Logic 5^th^ Ed*, however Professor Peter Suber, while at the Department of Philosophy, Earlham College (Richmond, Indiana) did compile solutions and a commentary to this and other chapters. Those that are relevant to this study unit are available at here. Although a couple of his solutions are not identical to those above, this does not mean that they are regarded as wrong, only different.