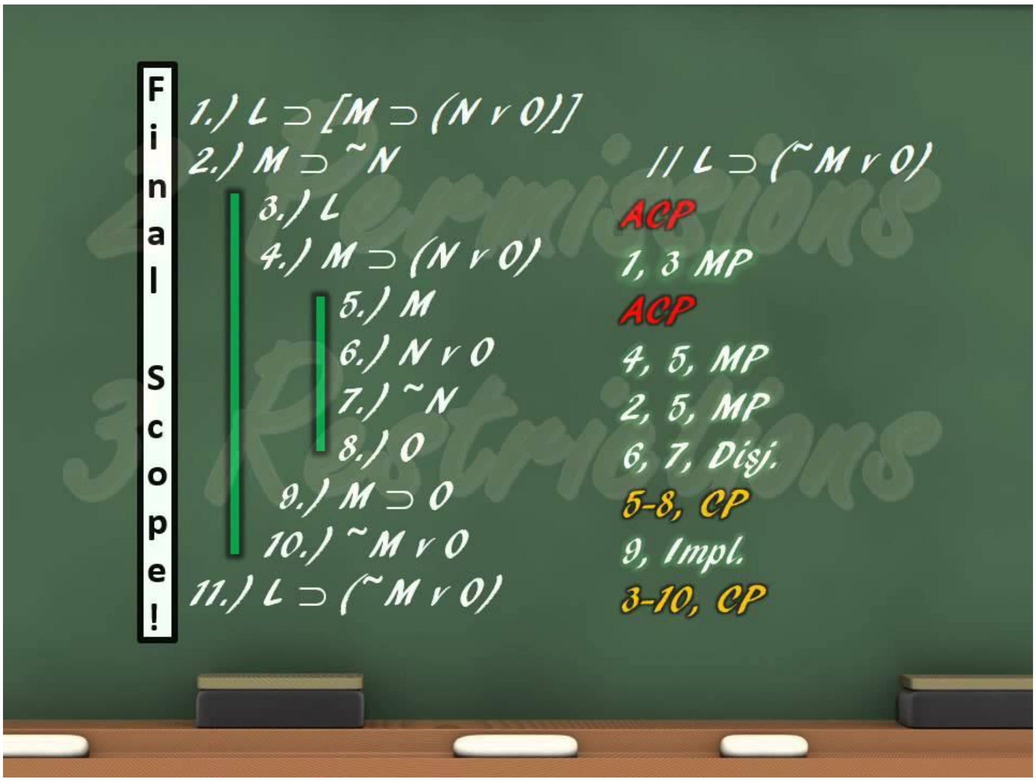
Critical Reasoning 09 - Conditional Proofs



Conditional proofs, such as the one on this chalk board, are a continuation of the method of deduction that we introduced in Critical Reasoning 07. A typical conditional proof takes the form of asserting a conditional, and proving that the antecedent of the conditional necessarily leads to the consequent. This method provides a way of proving valid arguments which cannot be otherwise proved valid by using only the 19 rules already listed. It also provides a way of shortening and simplifying certain proofs. In formal logic we are permitted, barring contradictions, to assume any antecedent to a conditional, so long can justify it or “cash it out” later by demonstrating that *if* the assumed antecedent *were* true then conclusion *would* necessarily follow. Note that the assumed antecedent or **conditional proof assumption** (**CPA**) as it is sometimes called, need not actually be true, only that if it *were* true it would lead to the consequent.

Suppose, for example, that we wanted to prove “A ⊃ C” from the premises:

1. A ⊃ B and
2. B ⊃ C

By now you will have recognised this as a hypothetical syllogism, so obviously we can’t use H.S. in our proof without assuming what we are trying to prove. If however, we assume that the antecedent “A” is true,

3. A CPA

then by conditional proof,

4. B 1,3 M.P.

5. C 2,4 M.P.
6. A ⊃ C 3-5 C.P.

Here "C.P." stands for "conditional proof" and represents the justification for our initial assumption. (Modified, Wikipedia : Conditional proof) Following the convention adopted by Copi (1979 p. 57 ff) we shall use a bent arrow with its head pointing at the assumption, with the vertical bit running along the length of the **scope** of the assumption and the horizontal bit marking off the end of the assumption, thus:

1. A ⊃ B and
2. B ⊃ C
3. A
4. B 1,3 M.P.
5. C 2,4 M.P.
6. A ⊃ C 3-5 C.P.

This notation is very useful for keeping track of multiple assumptions and their scopes in more complex proofs. When the scope of an assumption has been ended or closed off, the assumption is said to be **discharged**. The horizontal line should remind us that no subsequent line can be justified by reference to the assumption or by any line between it and the line inferred by the rule of conditional proof that discharges it. (Copi, 1979 p. 58) In other words we cannot subsequently appeal to what is enclosed by the arrow once it has been closed off.

E.g. 2.) Prove the following argument by the method of conditional proof using Copi's notation.

1. (M • N) ⊃ ~ P
2. P v Q
3. N /∴ M ⊃ Q

Hint: Begin by assuming M, thus:

1. (M • N) ⊃ ~ P
2. P v Q
3. N /∴ M ⊃ Q
4. M
5. M • N 4,3 Conj.
6. ~ P 1,6 M.P.
7. Q 2,6 D.S.
8. M ⊃ Q 4 -7 C.P.

The method of conditional proof also allows us to make multiple assumptions: one followed by another, having discharged the earlier assumption or by nesting assumptions, *i.e.* by make one assumption within the scope of another, as in the following example:

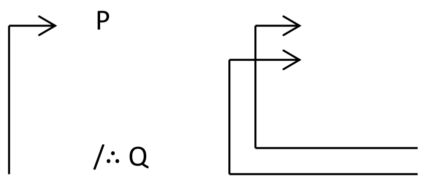
E.g. 3.) The following proof uses two nested assumptions.

1. A ⊃ (B ⊃ C)
2. B ⊃ (C ⊃ D) /∴ A ⊃ (B ⊃ D)

3. A
4. B
5. B ⊃ C 1,3 M.P.
6. C ⊃ D 2,4 M.P.
7. C 5,4 M.P.
8. D 6,7 M.P.
9. B ⊃ D 4-8 C.P.
10. A ⊃ (B ⊃ D) 3-9 C.P.

We could also have used H.S. here.

Note: The following diagrams represent incorrect uses of the method of conditional proof.



The proof on the left involves an assumption but has failed to discharge it. The one on the right has closed off a primary assumption while continuing to use the forgoing lines before closing off the secondary assumption. If the any of the lines in Copi's notation cross over in this way, then the condition proof in question has been incorrectly nested.

Justification: Perhaps by now you are satisfied that the method of conditional proof works in practice but are bit wary of the rationale of simply plucking assumptions from thin air, as it were. On the contrary, the work of a conditional proof, if one may call it that, consists in actually proving that the assumed antecedent of a conditional necessarily leads to its consequent. In fact, we can justify the method of conditional proof in terms concepts already familiar to us. Suppose that we begin with a generalised form of a conditional proof, assuming Q as an additional premise, thus:

P / ∴ Q ⊃ C
Q
..
..
..
C
Q ⊃ C C.P.

then,

1. as explained in Critical Reasoning 05, taking the premise "P" as the antecedent of a conditional and "Q ⊃ C" as the consequent, then argument "P / ∴ Q ⊃ C" corresponds to the truth function "*p* ⊃ (*q* ⊃ *c*)"
2. and by the rule of exportation: [*p* ⊃ (*q* ⊃ *c*)] ≡ [(*p* • *q*) ⊃ *c*].
3. Now, the truth function "(*p* • *q*) ⊃ *c*" correspond to the argument "P, Q /∴ C".
4. It follows that, if "P, Q /∴ C" is valid then "(*p* • *q*) ⊃ *c*" is a tautology, and
5. if "(*p* • *q*) ⊃ *c*" is a tautology, then "*p* ⊃ (*q* ⊃ *c*)" is also a tautology.
6. But if "*p* ⊃ (*q* ⊃ *c*)" is a tautology, then the argument P / ∴ Q ⊃ C is valid,

7. Therefore, we can prove that “P / ∴ Q ⊃ C” is valid by proving that “P, Q /∴ C” is valid.

Reductio ad absurdum (Latin: “reduction to absurdity” from the Greek: “εις άτοπον απαγωγή” or eis atopon apagoge, “reduction to the impossible”) is a common form of argument, know since antiquity and used both informally and formally in logic and mathematics, which seeks to demonstrate that a statement or propositions is true by showing that a false, untenable, or absurd result follows from its denial. (Wikipedia : Reductio ad absurdum)

Most readers will recall having used *reductio ad absurdum* in high school Euclidian Geometry proofs however the overall strategy is the same here: Assume that what you want to prove is false and see if it is possible to derive an inconsistency or contradiction from your assumption. If it is possible, then your original assumption that it was false must have been wrong and what you sought prove must be correct.

In Critical Reasoning 07 we introduced a technique whereby we were able to justify the validity of an argument by means of a corresponding truth table. However as we soon discovered the size of the truth table required grew exponentially (2^n) with the number of variables (n) involved. One way to circumvent this is to use the **shorter truth table technique** as a test of validity via the method of *reductio ad absurdum*. Recall that an argument of the form “P / ∴ C” is valid if the corresponding truth function “*p* ⊃ *q*” is a tautology and invalid if it is contingent (*i.e.* not a tautology.) Therefore:

1. If a contradiction results from the assumption, then “*p* ⊃ *q*” cannot be contingent; *i.e.* it must be a tautology and hence the argument “P / ∴ C” must be valid.
2. If no contradiction results from the assumption, then “*p* ⊃ *q*” must be contingent and the argument “P / ∴ C” must be invalid.

E.g. 4) Use the shorter truth table technique to test the validity of the following argument:

1. A ⊃ B
2. ∼ B /∴ ∼ A

Solution: Normally the shorter truth table technique is applied on a single line in one go, however for clarity we have shown all the intermediate steps below. First we write down the truth function that corresponds to this argument, thus:

[(A ⊃ B) • ∼ B)] ⊃ ∼ A

(If you are still unfamiliar with how to do so please referrer back to Critical Reasoning 05.) Next we assume that the truth function above is contingent, *i.e.* it is possible for it to be false. If so, then in a standard truth table, the value “F” will appear somewhere in the column of truth values under the operator with the largest scope; in this case “⊃”, thus:

[(A ⊃ B) • ∼ B)] ⊃ ∼ A
F

Next we attempt to fill in truth values under the remaining “columns” that would be consistent with this “F”. We know, for example, that the only time that “*p* ⊃ *q*” is false is when “*p*” is true and “*q*” is false, therefore we can deduce from our assumption that our truth function is consistent only under the following condition:

[(A ⊃ B) • ~ B)] ⊃ ~ A
T F F

And if "~ A" is false that would mean that "A" would have to be true, therefore we fill in the truth value, "T" under of all the "A's" in our truth function:

[(A ⊃ B) • ~ B)] ⊃ ~ A
T T F F T

Next we observe that "*p* • *q*" is true only when *p* and *q* are true together, therefore in this case both "(A ⊃ B)" and "~ B" must be true, thus:

[(A ⊃ B) • ~ B)] ⊃ ~ A
T T T T F F T

Now because "~ B" is true, that would mean that "B" would have to be false, therefore we fill in the truth value, "F" of all the "B's" in our truth function:

[(A ⊃ B) • ~ B)] ⊃ ~ A
T T F T T F F F T

When we have done so, we should notice a contradiction highlighted in red above: "A ⊃ B" can never be true when "A" is true and "B" is false. Because our assumption that the truth function is false leads to a contradiction that means that the truth function itself is a tautology and that the argument to which it corresponds is thus valid.

E.g. 5) Use the shorter truth table technique on a single line to test the validity of the same argument:

1. A ⊃ B
2. ~ B /∴ ~ A

Now that we are familiar with the step-by-step reasoning behind the shorter truth table technique, we can dispense with the wordy justification and simply fill in our initial assumption and the subsequent assignation of truth values required to make the corresponding truth function consistent, using numerals to indicate the order in which we do so. Therefore, anyone who understands the shorter truth table technique will then be able to follow our reasoning step-by-step. As above, we begin with the corresponding truth function and the initial assumption that it is contingent by placing an "F" and a ① under the operator with the largest scope, thus:

[(A ⊃ B) • ~ B)] ⊃ ~ A
F
①

Next we fill in the truth values required to make the assumption consistent beneath the operators and variables and number them as we proceed:

[(A	⊃	B)	•	~	B)]	⊃	~	A
T	T	F	T	T	F	F	F	T
③	④	⑤	②	④	⑤	①	②	③

For purposes of demonstration we have applied the shorter truth table technique over two lines, however for practical purposes we could have just written down the line above with a brief explanation as to why the highlighted contradiction arose and the implication for the validity of corresponding argument.

E.g. 6) Now use the shorter truth table technique on a single line to test the validity of the following argument for yourself:

1. A ⊃ ~ B
2. A v C /∴ B v C

Solution: As before the truth function to which this argument corresponds will be a conditional with the antecedent comprising of a conjunction of the premises and with the conclusion as the consequent, so:

[(A ⊃ ~ B) • (A v C)] ⊃ (B v C)

We can immediately make our *reductio ad absurdum* assumption that is false and proceed to fill in the truth values that would be consistent with that assumption.

[(A	⊃	~	B)	•	(A	v	C)]	⊃	(B	v	C)
T	T	T	F	T	T	T	F	F	F	F	F
⑤	③	⑥	④	②	⑤	③	④	①	④	②	④

Here we notice that we are able to fill in truth values consistent with the assumption that the truth function is false, therefore it is contingent and not a tautology. This means that the argument to which is corresponds is invalid.

The *reductio ad absurdum* method, and by extension the shorter truth table technique, is not appropriate to every argument form, however as Copi observes, "… in the vast majority of cases, [it] is superior to any other method known." (p. 62)

Proofs of Tautology and Contradiction: Most textbooks of formal logic have a section within the chapter on conditional proofs devoted to proofs of tautology and proofs of contradiction; however that would be superfluous here considering the techniques we have already mastered. A tautology is always true, irrespective of the truth values of its variables. A contradiction by contrast, is always false, irrespective of the truth values of its variables. Thus, "*p* ⊃ *p*" is a tautology because it is always true no matter what "*p*" and "*p* • ~ *p*" is a contradiction because it is always false no matter what "*p*", both of which can be tested using the method of *reductio ad absurdum*, which is itself a species of conditional proof. Any other truth function that is either not a tautology or a contradiction is simply contingent.

Tasks

a.) Construct conditional proofs for the following examples from Copi: (1979 p. 45)

21. S ⊃ (T • U)
(T v U) ⊃ V
∴ S ⊃ V

8. T ⊃ ~ (U ⊃ V)
∴ T ⊃ U

16. A ⊃ (B ⊃ C)
C ⊃ (D • E)
∴ A ⊃ (B ⊃ D)

b.) Use the shorter truth table technique to test the validity of the following arguments from Copi: (1979 p. 26):

1. If Alice is elected class president, then either Betty is elected vice-president or Carol is elected treasurer. Betty is elected vice-president. Therefore if Alice is elected class president, then Carol is not elected treasurer.

10. If Ed wins first prize, then either Fred wins second prize or George is disappointed. Fred does not win second prize. Therefore, if George is disappointed, then Ed does not win first prize.

13. If the weather is warm and the sky is clear, then we go swimming and we go boating. It is not the case that if the sky is clear, then we go swimming. Therefore the weather is not warm.

Feedback

a.) Conditional proofs:

21.

1.	S ⊃ (T • U)	
2.	(T v U) ⊃ V /∴ S ⊃ V	
→ 3.	S	
4.	T • U	1,2 M.P.
5.	T	4 Simp.
6.	T v U	5 Add.
7.	V	2,6 M.P.
8.	S ⊃ V	3-7 C.P.

8.

1. T ⊃ ~ (U ⊃ V) / ∴ T ⊃ U
2. T
3. ~ (U ⊃ V) 1,2 M.P.
4. ~ (~U v V) 3 Impl.
5. ~~U • ~V 4 DeM.
6. ~~U 5 Simp.
7. U
8. T ⊃ U 2-7 C.P.

16.

1. A ⊃ (B ⊃ C)
2. C ⊃ (D • E) / ∴ A ⊃ (B ⊃ D)
3. A
4. B
5. B ⊃ C 1,3 M.P.
6. C 5,4 M.P.
7. D • E 2,6 M.P.
8. D 7 Simp.
9. B ⊃ D 4-8 C.P.
10. A ⊃ (B ⊃ D) 3-9 C.P.

b.) Shorter truth table technique:

1. Let: A symbolise "Alice is elected class president," and
B: "Betty is elected vice-president" and
C: "Carol is elected treasurer," then this argument can be represented as:

A ⊃ (B v C)
B / ∴ A ⊃ ~ C

Now the truth function to which this argument corresponds is:

{[A ⊃ (B v C)] • B} ⊃ (A ⊃ ~ C)

Next we assume the operator with the largest scope to be false, thus:

{[A ⊃ (B v C)] • B} ⊃ (A ⊃ ~ C)
F
①

Next we proceed to fill in numbered truth valves consistent with our assumption:

{[A ⊃ (B v C)] • B} ⊃ (A ⊃ ~ C)
T T T T T T F T F F T
③⑤⑤④ ②⑤①③②③④

There is no contradiction in the assignment of truth values consistent with the assumption that this truth function is false. Therefore it is contingent and the argument to which it corresponds is invalid.

10. Let: E symbolise “Ed wins first prize,” and
F: “Fred wins second prize,” and
G: “George is disappointed,” then this argument can be represented as:

E ⊃ (F v G)
~ F / ∴ G ⊃ ~ E

The truth function to which this argument corresponds is:

{[E ⊃ (F v G)] • ~ F} ⊃ (G ⊃ ~ E)

Assuming that it is false and filling in numbered truth values consistent with our assumption, we get:

{[E ⊃ (F v G)] • ~ F} ⊃ (G ⊃ ~ E)
T T F T T T T F F T F F T
④⑧⑥⑦③ ②⑤⑥① ③②③④

Again, there is no contradiction in the assignment of truth values consistent with the assumption that this truth function is false. Therefore it is contingent and the argument to which it corresponds is invalid.

13. Let: W symbolise “The weather is warm,” and
C: “The sky is clear,” and
S: “We go swimming,” and
B: “We go boating,” then this argument can be represented as:

(W • C) ⊃ (S • B)
~ (C ⊃ S) / ∴ ~ W

The truth function to which this argument corresponds is:

{[(W • C) ⊃ (S • B)] • [~ (C ⊃ S)]} ⊃ (~ W)

Assuming that it is false and filling in numbered truth values consistent with our assumption, we get:

{[(W • C) ⊃ (S • B)] • [~ (C ⊃ S)]} ⊃ (~ W)
T T T T F F T T T F F F F T
③⑧⑥④⑥⑦ ②④⑥⑤⑥ ①②③

This time we have spotted a contradiction. The bracket on the left is true as shown by the T above ⑧ while the one immediately to the right of it is false as shown by the F above ⑦, yet the two brackets are joined by a “⊃” which has been marked true as shown by the T above ④. We know that a conditional can never be true with an antecedent that is true and a consequent that is false, hence the nature of the contradiction above. That makes this truth function a tautology, which is not possible to be false, and the argument to which it corresponds valid.

In the next critical reasoning study unit we take a look at probability from a non-specialist perspective.

Acknowledgement: Several of the examples above have been compiled from lecture notes on symbolic logic presented by the University of Cape Town.

Reference:

COPI, I.M. (1979) *Symbolic Logic,* 5th Edition. Macmillan : New York