Critical Reasoning 07 – The Method of Deduction

As we saw in Critical Reasoning unit 05, truth tables can be used to determine the truth-values and conditions of truth of potentially any complex statement; however the size of the corresponding truth tables becomes cumbersome beyond three or more components. Truth tables can also be used to justify the validity of potentially any deductive argument; however this method is beset with the same difficulty as the number of premises increases. Instead the **method of deduction** relies on a number of elementary arguments which are already known to be valid to *deduce* conclusions of more complex arguments from their premises.

We have already encountered nine such named elementary arguments or rules of inference in Critical Reasoning unit 02. Because by now you will be familiar with the use of symbols in logic, the elementary rules of inference can be reproduced in symbolic form with individual statements A, B, C and D replaced by variables *p*, *q*, *r* and *s* which can stand in for any statements, respectively.

Elementary Rules of Inference (Symbolic)

1. *Modus Ponens* (M.P.)
$p \supset q$
p
$\therefore q$

2. *Modus Tollens* (M.T.)
$p \supset q$
$\sim q$
$\therefore \sim p$

3. Hypothetical Syllogism (H.S.)
$p \supset q$
$q \supset r$
$\therefore p \supset r$

4. Disjunctive Syllogism (D.S)
$p \vee q$
$\sim p$
$\therefore q$

5. Constructive Dilemma (C.D.)
$(p \supset q) \bullet (r \supset s)$
$p \vee r$
$\therefore q \vee s$

6. Destructive Dilemma (D.D.)
$(p \supset q) \bullet (r \supset s)$
$\sim q \vee \sim s$
$\therefore \sim p \vee \sim r$

7. Simplification (Simp.)
$p \bullet q$
$\therefore p$

8. Conjunction (Conj.)
p
q
$\therefore p \bullet$ q

9. Addition (Add.)
p
$\therefore p \vee q$

Although anyone who understands the notation can *see* that each of these elementary rules of inference is valid, we can *justify* the validity of each by means of a truth table:

E.g. 1.) *Modus Ponens*: We begin by constructing a truth table with 2^nrows for n number of variables; in this case 2^2 = 4 rows for each of the permutations of T's and F's. Recall that a permutation is combination which takes into account the order in which things can be arranged. Next we create a column for each variable as well as extra columns for the premises and the conclusion. In the case of *modus ponens* the variables are *p* and *q* while the premises are $p \supset q$ and *p* respectively, with the conclusion q. Fortunately we do not have to duplicate any columns so that the truth table corresponding to *modus ponens* looks exactly like that defining the conditional $p \supset q$, thus:

p	q	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

Here our variables p and q are represented by columns 1 and 2, while our premises are represented by columns 3 and 1 respectively, with the conclusion q already represented by column 2. Now, according to our table it is only possible for the premises $p \supset q$ and p to be true together when q is also true, on row 1. Note that there is no row on our truth table on which both premises are true and the conclusion false. In other words it is not possible for $p \supset q$ to be false when p and q are both true, therefore the inference is valid.

E.g. 2.) *Modus Tollens*: Here again we require a truth table with 4 rows for the various permutations of the truth-values for p and for q. In fact we can use the same truth table as above but with 2 extra columns for $\sim p$ and for $\sim q$, thus:

p	q	$\sim p$	$\sim q$	$p \supset q$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Again our variables p and q are represented by columns 1 and 2, while our premises are represented by the last and second last columns respectively, with the conclusion $\sim p$ represented by column 3. According to our table it is only possible for the premises $p \supset q$ and $\sim q$ to be true together when $\sim p$ is also true, on the last row, or when p is false. Note that, as above, there is no row on our truth table on which both premises are true and the conclusion false, therefore the inference is valid.

E.g. 3.) Hypothetical Syllogism: Here we require a truth table with 2^3 rows to accommodate all possible permutations of truth-values for our three variables p, q and r. We will also require two extra columns for the premises $p \supset q$ and $q \supset r$ and one more for the conclusion $p \supset r$, as follows. Recall from Critical Reasoning 05 the distinctive T T T T F F F F, T T F F T T F F, T F T F T F T F pattern under the variable columns required to represent all the possible ordered arrangements of T's and F's in a three variable truth table.

p	q	r	$p \supset q$	$q \supset r$	$p \supset r$
T	T	T			
T	T	F			
T	F	T			
T	F	F			
F	T	T			
F	T	F			
F	F	T			
F	F	F			

Next we consult the truth table defining $p \supset q$ above to fill in truth-values under the conditionals. Recall that $p \supset q$ is true, except where p is true and q is false. The same applies to $q \supset r$ and $p \supset r$ respectively.

p	q	r	$p \supset q$	$q \supset r$	$p \supset r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Now, according to our table it is only possible for the premises $p \supset q$ and $q \supset r$ to be true together when $p \supset r$ is also true. Note that, as above, there is no row on our truth table on which both premises are true and the conclusion false, therefore the inference is valid.

We shall not proceed to prove the remaining six elementary rules of inference, save to mention that they can all be justified by truth tables in a similar fashion.

Formal Proofs of Validity

A **formal proof of validity** is a sequence of statements, each of which is either a premise of the argument being proved or follows directly from the preceding statements via an elementary valid argument, with the last statement in the sequence being the conclusion. One formal convention is to list the premises and the statements deduced from them in a single, numbered column with the justification for each deduced statement written alongside. Each justification should specify the number(s) of the previous statement(s,) as well as the elementary rule of influence by which the statement in question was deduced. For convenience we write the conclusion alongside the last premise separated by /∴ to mark it out as the conclusion for which we are aiming. (Copi, 1979 p. 33)

E.g. 4.) Formally prove the following argument: If it is a tree then it has roots. If it has roots then it absorbs minerals. It *is* a tree. Therefore it absorbs minerals.

The argument above may be symbolised, without difficulty, as follows:

T ⊃ R
R ⊃ M
T   /∴ M

It should be apparent that this argument involves an instance of hypothetical syllogism, however because H.S. here delivers the conditional conclusion T ⊃ M, we shall also require *modus ponens* to take us from T to M alone. A formal proof requires that we set out the method by which we deduce the conclusion via elementary rules of influence. The technique described by Copi may seem unnecessarily pedantic for such a trivial argument; however as the number of statements and instances of elementary rules of inference grow the proficiency of the technique to systematize large quantities logical content simultaneously becomes apparent. Therefore we begin by numbering and then listing the premises in a single column followed by /∴ and the conclusion alongside the last premise, thus:

1. T ⊃ R
2. R ⊃ M
3. T /∴ M

Next we observe that from lines 1 and 2 together we may infer T ⊃ M via a hypothetical syllogism, therefore we append this to our argument as line 4 and list its abbreviated justification alongside:

1. T ⊃ R
2. R ⊃ M
3. T /∴ M
4. T ⊃ M 1,2, H.S.

Now from lines 4 and 3 respectively we can infer the conclusion M via *modus ponens*; therefore we append that as line 5 together with its justification alongside. Note that the order in which we number the justifications is not arbitrary but should match the order of the elementary rules of inference cited.

1. T ⊃ R
2. R ⊃ M
3. T /∴ M
4. T ⊃ M 1,2, H.S.
5. M 4,3, M.P.

That is the end of our formal proof. We do not write *Q.E.D.* or draw attention to it in any other way. It is sufficient that we have *deduced* the conclusion from the premises in a systematic fashion, via a number of intermediary valid arguments.

E.g. 5.) Formally prove: 1. (H v J) ⊃ K
2. M v H
3. ~M /∴ K

The argument above is already in symbolic form and set out in the manner of the previous example. Note that lines 2 and 3 represent a disjunctive syllogism from which we may infer H, thus:

1. (H v J) ⊃ K
2. M v H
3. ~M /∴ K
4. H 2,3, D.S.

Now if H is true then H or any other statement is also true by addition. Some may feel that there is some sleight of hand in such an inference; however it may be justified by means of a truth table or illustrated by the following example: Think of any trivial statement that is true and any other statement whatsoever. Thus for example, if it is true that I own a blue sweater it is also true, without any trickery, that I own a blue sweater or I am the Empress Justine. Therefore we can append H v J as line 5 together with its justification, thus:

1. (H v J) ⊃ K
2. M v H
3. ~M /∴ K

4. H   2,3, D.S.
5. H v J   4, Add.

Now from lines 1 and 5 we may infer K by *modus ponens*, which completes our proof:

1. (H v J) ⊃ K
2. M v H
3. ~M /∴ K
4. H   2,3, D.S.
5. H v J   4, Add.
6. K   1,5, M.P.

E.g. 6.) Construct a formal proof for the following argument:

1. A v ~I
2. D ⊃ I
3. ~A
4. (~D • ~I) ⊃ W   /∴ W

Our penultimate aim in this argument should be to establish ~D • ~I because from there it is just one step of *modus ponens* away from our conclusion W; however since there is no way of deducing ~D • ~I directly from the premises we should try to deduce ~D and ~I separately and then conjoin them. From lines 1 and 3 we may infer ~I via a disjunctive syllogism, while from lines 2 and 5 we may infer ~D via *modus tollens*, thus:

1. A v ~I
2. D ⊃ I
3. ~A
4. (~D • ~I) ⊃ W   /∴ W
5. ~I   1,3, D.S.
6. ~D   2,5, M.T.

Next it is simply matter of conjoining ~D and ~I in that order to arrive at ~D • ~I from which we may infer W according to our strategy above, thus:

1. A v ~I
2. D ⊃ I
3. ~A
4. (~D • ~I) ⊃ W   /∴ W
5. ~I   1,3, D.S.
6. ~D   2,5, M.T.
7. ~D • ~I   6,5, Conj.
8. W   4,7, M.P.

Copi, 1979 p. 34 - 38 provides further exercises for practice.

Rules of replacement

While the method of deduction described so far is able to prove infinitely many arguments, there are some valid truth-functional arguments which cannot be proved by the nine elementary rules of inference alone. One such obviously valid argument is A • B /∴ B. Simplification only allows us to infer A • B /∴ A; therefore we are in need of an additional rule of inference that lets us infer B. That rule is commutation: $(p \bullet q) \equiv (q \bullet p)$. In ordinary language we do not bother with commutation because we intuitively understand that statements like 'Jack and Jill went up the hill' and 'Jill and Jack went up the hill' are equivalent, just as 1 + 2 and 2 + 1 are numerically equivalent. However, very early on, we learn of operations that do not commute, such as subtraction and division *e.g.* $1 - 2 \neq 2 - 1$. For that reason logicians and mathematicians are obliged to make steps involving commutation and other "obvious" rules of replacement, such as double negation, explicit in their formal proofs.

Unlike the first nine elementary rules of inference, which must be applied to a full line in a proof, the following ten rules of replacement allow the replacement of either a full line or part of a line in a proof with a logically equivalent expression, so that the truth-value of the resulting statement is the same as that of the original statement. We have already proved the first of these in Critical Reasoning 05.

Logically Equivalent Rules of Replacement

1. De Morgan's Theorems (De M.)	$\sim(p \bullet q) \equiv (\sim p \vee \sim q)$ $\sim(p \vee q) \equiv (\sim p \bullet \sim q)$
2. Commutation (Comm.)	$(p \bullet q) \equiv (q \bullet p)$ $(p \vee q) \equiv (q \vee p)$
3. Association (Assoc.)	$[p \bullet (q \bullet r)] \equiv [(p \bullet q) \bullet r]$ $[p \vee (q \vee r)] \equiv [(p \vee q) \vee r]$
4. Distribution (Dist.)	$[p \bullet (q \vee r)] \equiv [(p \bullet q) \vee (p \bullet r)]$ $[p \vee (q \bullet r)] \equiv [(p \vee q) \bullet (p \vee r)]$
5. Double Negation (D.N.)	$p \equiv \sim\sim p$
6. Transposition (Trans.)	$(p \supset q) \equiv (\sim q \supset \sim p)$
7. Material Implication (Impl.)	$(p \supset q) \equiv (\sim p \vee q)$
8. Material Equivalence (Equiv.)	$(p \equiv q) \equiv [(\mathrm{p} \supset \mathrm{q}) \bullet (q \supset p)]$ $(p \equiv q) \equiv [(p \bullet q) \vee (\sim p \bullet \sim q)$
9. Exportation (Exp.)	$[(p \bullet q) \supset r] \equiv [p \supset (q \supset r)]$
10. Tautology (Taut.)	$p \equiv (p \bullet p)$ $p \equiv (p \vee p)$

(after Copi, 1979 p. 39 - 40)

Rules of thumb and hints in the construction of formal proofs: Unlike truth tables, which can be churned out by mindless, automated process such as the use of a spread-sheet, formal proofs of validity require thought to figure out where to begin and how to proceed. Like Chess or Go there are certain moves that are permitted and others that are not; however there are no hard and fast rules of strategy. Similarly with formal proofs: what works is simply what is effective. Copi (p. 42) however does offer some suggestions. *Firstly*, try to simply deduce sub-conclusions from the premises via the nine rules of inference. These then become available as further premises by which we may reach the conclusion via subsequent rounds of deduction. *Secondly*, try eliminating statements that occur in the premises but not the conclusion. This can be done via simplification alone or commutation and then simplification. Similarly, a middle term *q* can be eliminated from the premises *p* ⊃ *q* and *q* ⊃ *r* via hypothetical syllogism to yield *p* ⊃ *r*. Distribution can also be used to transform the disjunction *p* v (*q* • *r*) into the conjunction (*p* v *q*) • (*p* v *r*). Either side can then be eliminated by either simplification alone or commutation and then simplification. *Thirdly*, try to introduce statements that are in the conclusion but not in the premises via addition. *Fourthly*, try to work backwards by looking for statements or pairs of statements from which the conclusion itself or a number of sub-conclusions can be deduced. None of these rules of thumb or hints is universally applicable, just as the advice to get your queen out early does not apply to every game of Chess. Typically they are used in combination with a thoughtful strategy in mind. The key, as always, is practice. To that end, consider the following examples compiled from lecture notes on symbolic logic presented by the University of Cape Town.

Replacement *vs.* Substitution

Replacement involves interchanging one statement for another that is logically equivalent to it, permitted by the ten rules of replacement above. Substitution, on the other hand, involves writing a statement in place of a variable in an argument with the same statement written in place of the same variable, if it occurs elsewhere in the argument. With replacement however, one or more statements may be interchanged with another that is logically equivalent to it without other occurrences of the same statement also having to be interchanged in the same way.

Formally prove the following arguments:

E.g. 7) 1. P ⊃ (M ⊃ N)
2. S ⊃ (P • M)
3. ~N /∴ ~S

Working backwards, we observe that we will be able to deduce ~S from ~N and S ⊃ N via *modus tollens* in our final line of proof, therefore our penultimate line should be S ⊃ N on its own. Therefore we look back to see how we might arrive at S ⊃ N from the given premises. We can't – not directly – however; the second premise S ⊃ (P • M) suggests the beginnings of a hypothetical syllogism, if only we could obtain (P • M) ⊃ N. Fortunately, exportation allows us to do just that because the first premise P ⊃ (M ⊃ N) is equivalent to (P • M) ⊃ N. Now we run the proof forward, recording each step with its justification alongside, thus:

1. P ⊃ (M ⊃ N)	
2. S ⊃ (P • M)	
3. ~N	/∴ ~S
4. (P • M) ⊃ N	1 Exp.
5. S ⊃ N	2,4, H.S
6. ~S	5,3, M.T

E.g. 8) 1. (A • A) ⊃ (R • S)
2. ~R v ~S /∴ ~A

One of the first things that leap out at us from this short argument is the unfamiliar A • A in the first line. At some stage we are going to have to replace it with just A using the replacement rule of tautology because none of our elementary rules of inference has tautological premises or conclusions. Next, given that we have ~A as the conclusion, we can deduce it from a tautology free version of the first premise and the negation of R • S on the right via *modus tollens*. Fortunately, ~(R • S) is logically equivalent to the second premise ~R v ~S according to De Morgan's theorem, therefore we have a strategy for our proof: Begin either with De Morgan's theorem on line 2 or tautological replacement on line 1, the order doesn't really matter so long as we do both, then conclude with ~A via *modus tollens* on the last line, thus:

1. (A • A) ⊃ (R • S)
2. ~R v ~S /∴ ~A
3. ~(R • S) 2 De M.
4. A ⊃ (R • S) 1 Taut.
5. ~A 4,3, M.T.

E.g. 9) 1. S ⊃ T
2. S v T /∴ T

Since this little argument comprises of only two terms, S and T, it could more economically be proved by a corresponding truth table of just four rows; however because it's formal proof requires multiple instances of replacement, it is instructive that we attempt to construct it here. Seeing that the conclusion is T alone, our strategy should be to isolate T by eliminating S or some variant of S as the middle term of a hypothetical syllogism. We cannot do so with premises as they stand, if only because we do not have the required three terms for S or some variant of S to stand in as a middle term. What we can do is replace them with equivalent expressions that have ~S in common, thus:

1. S ⊃ T
2. S v T /∴ T
3. ~T ⊃ ~S 1 Trans
4. ~~S v T 2 D.N.

Next, by implication we replace line 4 with an equivalent conditional which we can then use together with line 3. in a hypothetical syllogism.

1. S ⊃ T
2. S v T /∴ T
3. ~T ⊃ ~S 1 Trans
4. ~~S v T 2 D.N.
5. ~S ⊃ T 4 Imp.
6. ~T ⊃ T 3,5, H.S.

Now it is simply a matter deducing T alone from line 6. using implication and double negation, followed by tautology.

1. S ⊃ T
2. S v T /∴ T
3. ~T ⊃ ~S 1 Trans
4. ~~S v T 2 D.N.

5. ~S ⊃ T 4 Imp.
6. ~T ⊃ T 3,5, H.S.
7. ~~T v T 6 Imp.
8. T v T 7 D.N.
9. T 8 Taut.

Of course you will not be required to construct formal proofs as part of an introductory course in philosophy however, it is worth gaining some practice with them, even at this stage, as formal logic is usually a requirement to further study in philosophy at second and third year level. Given that you have come so far, you probably have more than just a passing interest in philosophy, therefore you may wish to attempt the following recommended tasks which require only a basic mastery of the method of deduction. At any rate they will help to consolidate what you have already learned.

Tasks

1. Construct a formal proof of validity for each of the following arguments using the recommended symbols. (Hint: no rules of replacement are required here.)

a.) If either algebra is required or geometry is required, then all students will study mathematics. Algebra is required and trigonometry is required. Therefore all students will study mathematics. (A, G, M, T)

b.)
1. (A v B) ⊃ ~D
2. A
3. D v E /∴ E

2. The following are formal proofs of validity. Annotate each line that is not a premise with its appropriate justification alongside.

a.)
1. (A • B) ⊃ [A ⊃ (D • E)]
2. (A • B) • C /∴ D v E
3. A • B
4. A ⊃ (D • E)
5. A
6. D • E
7. D
8. D v E

b.)
1. F v (G v H)
2. (G ⊃ I) • (H ⊃ J)
3. (I v J) ⊃ (F v H)
4. ~F /∴ H
5. G v H
6. I v J
7. F v H
8. H

3. Construct formal poofs of validity for the following arguments. (Hint: Rules of replacement will be required on several lines.)

a.) 1. A ⊃ ~(B v C)
2. (~B • ~C) ⊃ D
3. D ⊃ A
4. (A ≡ D) ⊃ X /∴ X

b.) 1. (P v B) ⊃ Z
2. ~Z
3. ~(D v X)
4. (~P • ~X) ⊃ (~R ≡ ~S) /∴ S ⊃ R

Feedback

1. Proofs:

a.) 1. (A v G) ⊃ M
2. A • T /∴ M
3. A 2 Simp.
4. A v G 3 Add.
5. M 1, 4, M.P.

b.) 1. (A v B) ⊃ ~D
2. A
3. D v E /∴ E
4. A v B 2. Add
5. ~D 1,4, M.P.
6. E 3,5, D.S.

2. Annotated proofs:

a.) 1. (A • B) ⊃ [A ⊃ (D • E)]
2. (A • B) • C /∴ D v E
3. A • B 2 Simp.
4. A ⊃ (D • E) 1,3, M.P.
5. A 3 Simp.
6. D • E 4,5, M.P.
7. D 6 Simp.
8. D v E 7 Add.

b.) 1. F v (G v H)
2. (G ⊃ I) • (H ⊃ J)
3. (I v J) ⊃ (F v H)
4. ~F /∴ H
5. G v H 1,4, D.S
6. I v J 2,5, C.D.
7. F v H 3,6, M.P.
8. H 7,4, D.S.

3. Proofs requiring rules of replacement:

a.)

1. A ⊃ ~(B v C)
2. (~B • ~C) ⊃ D
3. D ⊃ A
4. (A ≡ D) ⊃ X /∴ X
5. A ⊃ (~B • ~C) 1 De M.
6. A ⊃ D 5,2, H.S.
7. (A ⊃ D) • (D ⊃ A) 6,3, Conj.
8. A ≡ D 7 Equiv.
9 X 4,8, M.P.

b.)

1. (P v B) ⊃ Z
2. ~Z
3. ~(D v X)
4. (~P • ~X) ⊃ (~R ≡ ~S) /∴ S ⊃ R
5. ~(P v B) 1,2,M.T.
6. ~P • ~B 5 De M.
7. ~D • ~X 3 De M.
8. ~P 6 Simp.
9. ~X • ~D 7 Comm.
10. ~X 9 Simp.
11. ~P • ~X 8,10, Conj.
12. ~R ≡ ~S 4,11, M.P
13. (~R ⊃ ~S) • (~S ⊃ ~R) 12 Equiv.
14. ~R ⊃ ~S 13 Simp.
15. S ⊃ R 14 Trans.

If you had no difficulty with the above tasks then you are likely to cope well with similar material in Symbolic Logic, which forms part of the middle to senior curriculum of Philosophy. If on the other hand, you had trouble constructing proofs but did have an “a-ha” moment of insight when you worked through the solutions, then you have achieved what this study unit set out to accomplish, namely to familiarise yourself with the basic method of deduction. Otherwise, do not despair. You are no less likely to recognise or come up with a valid argument of your own when required, even though you might not necessarily be able to furnish a formal proof. Besides, there are other methods of demonstrating validity such as truth tables and Venn diagrams which are reliable, all be they cumbersome when many statements are involved.

The next Critical Reasoning unit describes the phenomenon of “groupthink,” while the one after that returns to the method of deduction using conditional proofs and the method of *reductio ad absurdum*: Latin for “reduction to absurdity”

Reference:

COPI, I.M. (1979) *Symbolic Logic,* 5th Edition. Macmillan: New York