Critical Reasoning 05 – Complex Statements

A complex system can only be understood if the components can be identified.

The statements that we have been considering so far have all been of ‘the cat is on the mat,’ or ‘Socrates is mortal’, hum-drum variety. Such simple statements do not comprise of other statements. Complex statements by contrast incorporate other statements as part of themselves. The complex statements used in the argument by Socrates in Plato’s *Apology* that we encountered in Critical Reasoning 01 are more typical of those that we confront in real world philosophical texts. Fortunately, they can all be understood if we can identify and grasp their component parts and the conjunctions that string them together.

In the same way that an infinite number of arithmetic expressions can be built up using just a handful of operations (addition and subtraction), similarly an infinite number of complex statements can be built up with the help of just four **logical connectives** (‘and,’ ‘or,’ ‘not’ and ‘if… then.’) In logic, such connectors mean pretty much what they do as conjunctives in plain language, though stripped of their nuances but retaining a surprising degree of regulation. As in arithmetic there are standard symbols for each, and standard ways of building up complex sentences using them.

Before we begin, there are a couple of very important conventions to follow. We need to distinguish between actually using a term or connective as opposed to just mentioning one. Thus:

Katrina was a deadly hurricane (use)

vs.

‘Katrina’ is the name of a hurricane (mention)

Terms merely mentioned should be placed between single inverted commas as this avoids potential ambiguity and confusion, especially in complex expressions. When quoting text directly, such as Anne Frank’s, “No one has ever become poor by giving” we use double inverted commas. The only other time we use double inverted commas is when we wish to draw attention to something or some concept that is ironic, sarcastic, inappropriate or just plain wrong. These are known as scare quotes. *E.g.* Columbus “discovered” America. Often in speech we make “air quotes” with our fingers in order to signify this meaning.

The logical connectives and their symbolisation are set out below. Note that capital letters are used to represent simple sentences just as x's and y's are used to represent variables in algebra. Don't panic if you have an aversion to algebra - this is not High School Maths all over again! As a thinking, rational human being you already understand and use the fundamentals of basic logic every day, it is just getting them down on paper that takes some getting used to.

A *and* B symbolised as A • B means that both statement A and statement B have to be true for the **conjunction** A • B to be true. In other words if either A or B or both turn out to be false then so does the conjunction A • B. The logical connector 'and' also covers the English conjunction 'but' which means the same as 'and' with a contrast. *E.g.* A tale that is "strange *but* true" is only so if it is both strange *and* true.

A *or* B symbolised as A v B means that either statement A or B or both A and B have to be true for the **disjunction** A v B to be true. In other words, if both A and B turn out to be false then so does the disjunction A v B. There is another sense in which we sometimes use 'or' that is exclusive, such as in a game of tennis where it is true that A *exclusive or* B will win. Either A will win or B will win but both of them cannot win together. Similarly both of them cannot *not* win together. A *exclusive or* B symbolised as **A XOR B** is true when either statement A or B is true but not when both statements A and B are simultaneously true or simultaneously false. Alternatively, one can remember that A XOR B is true only when A and B have different truth values. The Romans had separate words for 'inclusive or' (*vel*) *vs.* 'exclusive or' (*aut*). We will use the inclusive 'or' ahead; however the choice is a matter of convention. XOR was the traditional preference; however even today electrical engineers and computer programmers, for example, frequently construct XOR logic gates in their circuitry or use XOR operators in their programs respectively.

Not A symbolised as ∼A means that A is negated. Whatever the truth or falsity of A, ∼A just flips the truth value to its opposite. In the same way that a coin flipped from heads to tails once will be flipped back to heads a second time, so the **double negation** of any statement just gives you the original statement back. *E.g.* 'It not true that Lance does not cheat' (∼∼L) is the same as 'Lance cheats' (L). There are some uses of double negation in the vernacular where the intention is to emphasise the negation by repeating it such as, 'I ain't got nobody' or 'He don't know nothing.' Then there are those languages, such as Afrikaans, where double negation is the grammatical but not semantic norm, rather than the exception.

If A… then B symbolised as A ⊃ B is used to express **material implication** or a **conditional** or **hypothetical**. (Note: we use the term material implication here to distinguish it from other kinds of implication, such as logical or causal implication.) Here A is referred to as the ground, **antecedent** or protasis, while B is referred to as the **consequent** or apodosis. Any conditional is true when the both the antecedent and the consequent are true but false when the antecedent is true and the consequent is false. That much should be clear from the meaning of a conditional in plain language. However we should also note that just because conditionals and conjunctives are true when both A and B are true and false when A is true and B is false, what might not be so obvious in plain language is that conditionals are also true whenever the antecedent is false, irrespective of the truth or falsity of the consequent. This will become apparent when we encounter the formal definition of A ⊃ B.

Only clarification: This little word is used to designate special cases; however there are at least three very different conditionals in which it can be employed.

> **A only if B** is a special case of A implies B therefore it is also symbolized as A ⊃ B. *E.g.* 'Patrons will be admitted to the concert *only if* they produce a ticket,' implies that '*if* a patron produces a ticket *then* he or she will be admitted to the concert.'
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> **Only if A then B** is symbolised as B ⊃ A, which looks a little counterintuitive; however think of it this way: if the only time that B ever occurs is when A does, then you can deduce that B's occurrence implies A's. *E.g.* '*Only if* there is oxygen present (*then*) a match will burn,' implies that '*if* a match is burning *then* there is oxygen present.'
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> **A if and only if B** is symbolised as (A ⊃ B) • (B ⊃ A) because the expression implies that two situations or entities are effectively materially equivalent. *E.g.* 'I will receive a bonus from my boss *if and only if* I complete one hundred tasks,' implies that 'my boss will reward me with a bonus if I complete one hundred tasks,' and 'if I complete one hundred tasks my boss will reward me with a bonus.' Because the first implies the second and the second implies the first we have an implication going both ways or a **biconditional**, hence: (A ⊃ B) • (B ⊃ A), which is contracted to A ≡ B. Read: 'A is materially equivalent to B.' In logic and mathematics A *if and only if* B is often abbreviated as A *iff* B

Parentheses (brackets) symbolised as (…) or […]or {…} are used in logic as they are in Mathematics, Physics or Chemistry to indicate grouping or to signify the **order of precedence** of operators, such as (A • B) ⊃ C with one level of nesting, or ∼ ((A • B) ⊃ C) with two, or others with even more levels of nesting in which the innermost parentheses takes precedence.

Truth tables are diagrams in rows and columns showing how the truth or falsity of a proposition varies with that of its components. To set up a truth table for a complex expression, you first need to count the number of simple sentences or propositions (n) that go to make up that expression. You will need 2^nrows on paper (or a spread sheet) to accommodate all the possible **permutations** (see box below) of true and false instances of simple sentences or propositions on your table. Next you will require a column for each simple sentence or proposition and one more for the final expression plus an extra column for every extra degree of nesting in the finals expression. We can define the logical connectives themselves by their truth tables:

Ludwig Wittgenstein (1889 - 1951) Austrian born philosopher of logic and language and inventor of the truth table: pictured 1947

> **A and B** requires $2^n = 4$ rows to accommodate all the permutations of true and false. It also requires three columns: one for A, one for B and one for the compound expression A • B. So first, we label the table and draw up the columns as follows.

and

A	B	A • B

Now we fill in the first two columns with every permutation of T's and F's. If we do this systematically, as below, we will cover every permutation without duplicating or leaving any out. So now, our partial truth table for 'and' looks like this:

and

A	B	A • B
T	T	
T	F	
F	T	
F	F	

Combinations *vs.* Permutations

A combination refers to the number of ways of arranging items period. A permutation however is a combination that takes the order of sequence of items into account. For some logical connectives like 'and' and 'or' the order is irrelevant, however for others such as 'if… then…' switching the order of what comes after 'if…' with what comes after 'then…' can reverse the truth value of an expression. So if order matters, think permutation rather than combination.

Next we fill in the truth values under A • B. Since we know that A and B is true only when both A and B are simultaneously true and false otherwise, we can complete the truth table for 'and' as follows:

and

A	B	A • B
T	T	T
T	F	F
F	T	F
F	F	F

We can now see how 'and' is *defined* by its truth table. Following the same procedure we can draw up truth tables for 'or' and 'not'.

or

A	B	A v B
T	T	T
T	F	T
F	T	T
F	F	F

not

A	~A
T	F
F	T

If… then… is symbolised by A ⊃ B and because a conditional cannot be true when the antecedent is true and the consequent is false, A ⊃ B can be negatively defined as: not (antecedent and not consequent) or ~(A • ~B) which can be tabulated as:

A	B	~B	A • ~B	~(A • ~B)	A ⊃ B
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Therefore 'if… then…' is also *defined* by its own special truth table:

A	B	A ⊃ B
T	T	T
T	F	F
F	T	T
F	F	T

Note the distinctive T F T T pattern under the A ⊃ B column which shows that if A then B is true when both the antecedent and consequent are true, and false when the antecedent is true and the consequent is false. It also shows that if the antecedent is false then, irrespective of the truth of the consequent, the conditional will be true. This has nothing to do with the English meaning of the conditional but is a consequence of its definition as 'not (A and not B)'. See the table at the top of this page, second column from the right.

Truth tables can be used to determine the truth or falsity of any compound statement based on the truth or falsity of its simple components. See if you can predict the truth or falsity of the following compound statements based on the truth or falsity of their constituent statements and the function(s) of their logical connectives. It is invariably helpful to symbolise more lengthy statements because that way they can be jotted down quickly and taken in at a single glance. (These examples are drawn (or adapted) from Copi, 1979 p.15 & 19. The first three are Biblical in origin.)

1. The words of his mouth were smoother than butter, but war was in his heart.

   Let "The words of his mouth were smoother than butter" = M and "War was in his heart" = W, then the compound statement can be symbolised as: M • W. Because only two simple conjunct statements are involved, we can read off their compound truth values directly from the truth table for 'and'. Therefore this statement will be true only when both M and W are simultaneously true, but false otherwise.

2. Promotion cometh neither from the East, nor from the West, nor yet from the South.

   Let "Promotion cometh from the East" = E and "Promotion cometh from the West" = W and "Promotion cometh from the South" = S, then because 'neither…nor…' is equivalent to 'not…and not...', the compound statement can be symbolised as: ~E • ~W • ~S. Here again there is no need to draw up a truth table when can simply read off the compound truth values from the conjunction of negated statements. Because 'and' is true only when all its components are true and 'not' flips the truth value of anything it negates, this compound statement will be true only when E, W and S are simultaneously false.

3. As for man, his days are as grass: as a flower of the field, so he florisheth.

   Let "Man's days are as grass" = D and "Man florisheth as a flower of the field" = F, then even though the word 'and' is not used, it is implied, therefore the compound statement can be symbolised as: M • F. So for the same reason as 1 above, this statement will be true only when both M and F are simultaneously true, but false otherwise.

4. If Argentina wins its first game, then both Congo and Denmark win their first games.

   Let "Argentina wins its first game" = A and "Congo wins its first game" = C and "Denmark wins its first game" = D, then we need to bracket the consequent of this conditional because it has two parts, namely: C and D. Therefore the compound statement is symbolised as: A ⊃ (C • D). The quick way to figure out the truth conditions of this statement is to look up or remember the conditions for 'A ⊃ B' and then substitute B for C • D. The longer, but fool proof, method is to construct a truth table with columns for the components and enough rows to ensure we have every permutation of T and F covered; in this case 2^3 for three variables = 8, thus:

A	C	D	C • D	A ⊃ (C • D)
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

Now we fill in the truth values under the C • D and A ⊃ (C • D) columns according to the truth tables for 'and' and 'if…then…' respectively.

A	C	D	C • D	A ⊃ (C • D)
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

So from the truth table above, we can see that A ⊃ (C • D) is true when A, C and D are simultaneously true or whenever A is false, irrespective of the truth-value of C or D.

5. Only if both Argentina and Congo win their first games will Denmark not win its first game.

   Again let "Argentina wins its first game" = A and "Congo wins its first game" = C and "Denmark wins its first game" = D. The appearance of 'only if…then…' should remind us that the direction of implication is reversed in such statements. Remember, 'only if oxygen is present, then a match will burn' is equivalent to B ⊃ A. So here: not D implies (A and C) is symbolised as ~D ⊃ (A • C). Again we will have to construct a truth table with $2^3 = 8$ rows to cover all the permutations of true and false, plus 3 extra columns to capture the 'not', the "hook" (⊃) and the nested 'and' as follows:

A	C	D	~D	(A • C)	~D ⊃ (A • C)
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	T	F	F
F	T	T	F	F	T
F	T	F	T	F	F
F	F	T	F	F	T
F	F	F	T	F	F

   So from the truth table above, we can see that ~D ⊃ (A • C) is true, except when D is false and both A and C are false.

6. If Argentina does not win its first game, then it is not the case that either Congo or Denmark wins its first game.

   Again let "Argentina wins its first game" = A and "Congo wins its first game" = C and "Denmark wins its first game" = D. This statement involves two negated statements (the second one a compound,) involved in a conditional *i.e.* not A implies not (C or D) or simply 'A ⊃ ~(C v D). Now we construct a truth table for three variables and populate it with T's and F's in the usual manner:

A	C	D	~A	(C v D)	~(C v D)	~A ⊃ ~(C v D)
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	F	T	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	T	F	T	T	F	F
F	F	T	T	T	F	F
F	F	F	T	F	T	T

   So from the truth table above we can see that ~A ⊃ ~(C v D) is true, except when A is false and C or D are true.

Applications: Although the examples above are highly contrived and have been modified for didactic purposes, it should be clear that it is possible to deduce the conditions under which any complex statement might be true or false, respectively. Therefore, researchers can determine, in advance, which permutations of variables should be examined or determined by experiment to be either true or false, rather testing every single possible permutation, which can be time consuming and costly, especially when there are many variables involved. Truth tables also allow philosophers, logicians and mathematicians to test and prove prospective theorems.

A **theorem** is a statement, in words or symbols, that is formally true (or **tautological**,) in which all substitution instances are true. A **contradiction**, on the other hand, has only false substitution instances. *E.g.* We can use a truth table to prove De Morgan's Theorem, which states that the negation of a conjunction of two statements is equivalent to the disjunction of their negation. In symbols: $\sim(p \bullet q) \equiv (\sim p \vee \sim q)$. Accordingly, saying that it is not the case that Jack and Jill ran up the hill is equivalent to saying that either Jack did not run up the hill or Jill did not run up the hill. Recall that we have already encountered the meaning of 'equivalence' under '… if and only if…'. There we noted that p is equivalent to q is symbolised as the biconditional:

Augustus De Morgan (1806-1871) British Mathematician and Logician

$(p \supset q) \bullet (q \supset p)$ or simply $p \equiv q$

So if we construct a truth table for $(p \supset q) \bullet (q \supset p)$ we get the combination of truth values that define $p \equiv q$ thus,

$(p \supset q) \bullet (q \supset p)$

p	q	$(p \supset q)$	$(q \supset p)$	$(p \supset q) \bullet (q \supset p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

which defines:

$p \equiv q$

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

Now we simply draw up a truth table for De Morgan's Theorem $\sim(p \bullet q) \equiv (\sim p \vee \sim q'$ and note what truth values appear under the equivalence (≡) column. If they are all true then the statement is formally true and hence a theorem. If they are all false then the statement is a contradiction and thus formally false, otherwise it is neither.

p	q	$(p \bullet q)$	$\sim(p \bullet q)$	$\sim p$	$\sim q$	$(\sim p \vee \sim q)$	$\sim(p \bullet q) \equiv (\sim p \vee \sim q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Therefore De Morgan's Theorem is a theorem indeed! It is unlikely however that you will be asked to produce any formal proofs at first year level, but because second and third year courses dealing with symbolic logic and the philosophy of language draw heavily on formal arguments, it worth getting a taste of them here.

If you have been able to follow the examples carefully, you will be prepared for the second to next topic on argument forms and the method of deduction. If you feel you still need practice symbolising complex statements and deciding when they are true or false, you might want to try some of the exercises in Copi (1979, p. 18-19).

Task

Prove that the following rules of replacement are theorems by means of a truth table for equivalence:

1. Double Negation: $p \equiv \sim\sim p$
2. Commutation: $(p \vee q) \equiv (q \vee p)$
3. Transposition: $(p \supset q) \equiv (\sim q \supset \sim p)$
4. Material Implication: $(p \supset q) \equiv (\sim p \vee q)$
5. Exportation: $[(p \bullet q) \supset r] \equiv [p \supset (q \supset r)]$

Hint: Recall that $p \equiv q$ is true when both p and q are true together or false together and false when p is true and q is false or *vice versa* and that a theorem can be shown to be a tautology, *i.e.* one in which all substitution instances are true.

Feedback

The equivalence truth tables by means of which the above theorems may be proved are as follows:

1.)

p	$\sim p$	$\sim\sim p$	$p \equiv \sim\sim p$
T	F	T	T
F	T	F	T

Since all the substitution instances under the last column are true, $p \equiv \sim\sim p$ is a tautology and hence a theorem. The same is true of the following:

2.)

p	q	$p \vee q$	$q \vee p$	$(p \vee q) \equiv (q \vee p)$
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

3.)

p	q	$\sim p$	$\sim q$	$p \supset q$	$\sim q \supset \sim p$	$(p \supset q) \equiv (\sim q \supset \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

4.

p	q	$\sim p$	$p \supset q$	$\sim p \vee q$	$(p \supset q) \equiv (\sim p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

5.

p	q	r	$p \bullet q$	$q \supset r$	$(p \bullet q) \supset r$	$p \supset (q \supset r)$ …
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	T
F	T	F	F	F	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

$[(p \bullet q) \supset r] \equiv [p \supset (q \supset r)]$
T
T
T
T
T
T
T
T

Rules of replacement, such as those above, can be used to manipulate propositions for use in formal proofs, without changing their truth conditions: more of which in Critical Reasoning 07. The next critical reasoning unit however concerns heuristics which are mental shortcuts and rules of thumb in problem solving.

Reference:

COPI, I.M. (1979) *Symbolic Logic*, 5th Edition. Macmillan: New York